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Computability on Regular Subsets of Euclidean Space

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Abstract. For the computability of subsets of real numbers, several reasonable notions have been suggested in the literature. We compare these notions in a systematic way by relating them to pairs of 'basic' ones. They turn out to coincide for full-dimensional convex sets; but on the more general class of regular sets, they reveal rather interesting 'weaker/stronger' relations. This is in contrast to single real numbers and vectors where all 'reasonable' notions coincide.

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1 Introduction

When should we call a real number $r \in \mathbb{R}$ '*computable*'? Starting with Alan Turing's seminal work [21] several intuitive definitions have been investigated in the literature most of which turned out to be equivalent [4, 23, 16, 19, 1, 22] for single real numbers as well as for vectors (tuples) and functions [9, 17, 13, 22].

For infinite subsets, the situation is not quite the same: Investigating on the computability of geometric problems such as convex hull and linear optimization, several authors have treated various useful notions of computability [8, 14, 10, 6, 11, 3, 24].

Ge and Nerode [8] for instance call a (closed non-empty) set $A \subseteq \mathbb{R}^d$ *computable* iff its Euclidean distance function

$$(1) \qquad d_A \;:\; \mathbb{R}^d \ni x \;\mapsto\; \inf\big\{ \|x-a\|_2 \;:\; a \in A \big\} \;\in\; \mathbb{R}, \qquad \|y\|_2^2 = \sum y_i^2$$

is a computable function on reals; whereas Grötschel, Lovász, and Schrijver [10] consider a full-dimensional compact convex set $K \subseteq \mathbb{R}^d$ *computable* iff some Turing machine can solve the Weak Membership Problem representing K, that is,

for every $y \in \mathbb{Q}^d$ and rational $\delta > 0$ either (i) assert that $\overline{B}(y,\delta) \cap K \neq \emptyset$ [1)] or (ii) assert that $\overline{B}(y,\delta) \setminus K \neq \emptyset$;

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One more example, Edalat and Lieutier in their domain theoretical setting [6] represent a regular set R in d-space by a pair (U, V) of open subsets such that $U \subseteq R \subseteq \mathbb{R}^d \setminus V$ and $\overline{U} \cup \overline{V} = \mathbb{R}^d$; they consider both U and V given as unions over a countable topological base such as the open rational balls $\{B(x,r) : x \in \mathbb{Q}^d, r \in \mathbb{Q}_+\}$.

So how are these notions of computability related to each other? A thorough treatment and comparison for the case of closed sets in Euclidean space can be found in [3] with generalizations to metric spaces in [2]. For full-dimensional compact convex subsets of $\mathbb{R}^2$, [14] compares several notions and finds them non-uniformly equivalent.

The present work focuses on *regular* sets of Euclidean space; these lie between the (in our opinion too general) closed ones and the (too restrictive) closed full-dimensional bounded convex ones. Roughly speaking, regularity of a set maintains the full-dimensionality requirement (no lines or points) while dropping the boundedness condition as well as convexity and even connectedness.

We introduce ten rather natural 'basic' ways of encoding regular subsets and show that each of the previously mentioned notions of computability is uniformly equivalent to a pair of them. Relations between them, such as 'stronger' or 'weaker' can then be easily read off from this paper's central result: a systematic investigation of the ten basic encodings' properties and convertability among them. For the case of *convex* regular sets, the previous notions turn out to be uniformly equivalent. This generalizes work of Kummer/Schäfer who needed boundedness of their sets, required dimension 2, and considered only the non-uniform case.

2 Computability of Euclidean Subsets

We refer to Weihrauch's *Type-2 Theory of Effectivity* (TTE) as described in [22] and will deliberately make use of this convenient framework and its handy tools for comparing notions of computability and for constructing new from given ones by means of so-called *representations*. Roughly speaking, a representation of some universe U is an assignment of infinite strings ('*names*') $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in \Sigma^{\mathbb{N}}$ to all of its members $u \in U$. Such u will be called *computable* (with respect to that representation) iff some Turing Machine is capable of generating, during infinite process of operation, a name of u. When α and β are representations for universes U and V, respectively, a function $f : U \to V$ is (α, β)-*computable* if some Turing Machine is capable of, upon input of any α-names for any $u \in U$, generating a β-name for $v = f(u)$.

In the next section we will also need the *join*, a very basic operation on representations. Consider representations α and β for the same set U. The new representation $\alpha \sqcap \beta$ for U then supplies, for each $u \in U$, two names as follows: Take an α-name $\bar{\sigma}$ and a β-name $\bar{\tau}$ for $u \in U$; merge these two infinite strings in a zipper-like way: this resulting string $(\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_n, \tau_n, \ldots)$ will be an $\alpha \sqcap \beta$-name for u.

Comparisons between two representations α and β for U (and their induced notions of computability) are performed in two ways: If every α-computable $u \in U$ is β-computable, too, we will write "$\alpha \Rightarrow \beta$". Notice the inherent *non-uniformity*! Indeed, "$\alpha \Rightarrow \beta$" means that, to any Turing Machine $M_{\alpha,u}$ generating an α-name for u

1)Here and in the sequel, let $B(y, \delta) \subseteq \mathbb{R}^d$ denote the open Euclidean ball with radius $\delta > 0$ and center $y \in \mathbb{R}^d$, $\bar{S}$ the topological closure of $S \subseteq \mathbb{R}^d$ and S° its topological interior.

there exists a Turing Machine $M_{\beta,u}$ which outputs a β-name for the same u: nothing is said about the effectivity of this transformation $M_{\alpha,u} \to M_{\beta,u}$. If, however, one single Turing Machine is capable of converting *any* α-name into some corresponding β-names, this *uniform reducibility* will be written as "$\alpha \preccurlyeq \beta$"; similarly, "$\alpha \Leftrightarrow \beta$" for non-uniform *equivalence* and "$\alpha \equiv \beta$" for its uniform counterpart.

TTE also considers a further, slightly weaker notion called *continuous reducibility*, "$\alpha \preccurlyeq_t \beta$". We remark that our uniform results hold for this notion as well. In fact, all non-reducibility proofs in the present work argue that presumed reducibility must necessarily be discontinuous. The only exception can be found in Theorem 4.10g).

For the universe (hyperspace) $\mathcal{A}^d$ of all[2)] closed subsets $A \subseteq \mathbb{R}^d$, a ψ^d-name for A is an encoding of its distance function $d_A : \mathbb{R}^d \to \mathbb{R}$ as defined in Equation (1). Such a name permits to, upon input of any point $x \in \mathbb{R}^d$, effectively approximate x's distance to the set A up to arbitrary precision. Similarly, a $\psi^d_>$-name permits to approximate distances from *below*, a $\psi^d_<$-name from *above*, cf. [3]. There it was also shown that $\psi^d_>$ encodes negative information about sets whereas $\psi^d_<$ supplies positive information, compare Exercise 5.1.13 in [22].

Representations of open subsets $U \subseteq \mathbb{R}^d$ are readily obtained by considering their complements as these are closed: A $\theta^d_>$-name for U is a $\psi^d_<$-name for $A = \mathbb{R}^d \setminus U$; and a $\theta^d_<$-name for U is a $\psi^d_>$-name for A. Again, $\theta^d_>$ encodes negative information and $\theta^d_<$ positive one. It is well known [3] that these encodings induce (both uniformly and non-uniformly) independent notions of computability, that is:

$$\theta^d_< \not\Rightarrow \theta^d_>, \qquad \theta^d_> \not\Rightarrow \theta^d_<, \qquad \psi^d_< \not\Rightarrow \psi^d_>, \qquad \psi^d_> \not\Rightarrow \psi^d_< \ .$$

3 Regular Subsets

We emphasize that each of the above representations are straightforward ways of encoding sets and have relevance to practical applications; cf. pp.127-128 in [22].

However, as already indicated in the introduction, the class of closed (and correspondingly that of open) Euclidean subsets might be too general for some purposes. This is reflected by the fact that for example intersection $\cap : \mathcal{A}^d \times \mathcal{A}^d \to \mathcal{A}^d$ is *not* $\big((\psi^d_< \sqcap \psi^d_>) \times (\psi^d_< \sqcap \psi^d_>), \psi^d_<\big)$-computable. In other words: one cannot effectively obtain positive information about the result $A \cap B$ upon input of A's and B's respective $\psi^d_< \sqcap \psi^d_>$-names [3].

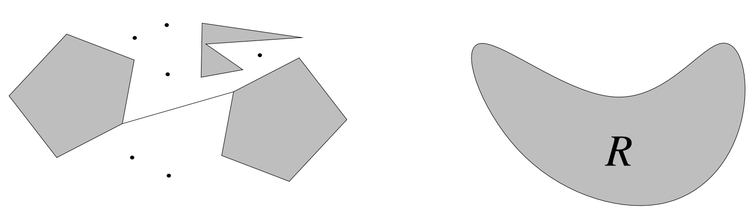

Figure 1: a non-regular set and a regular set R.

2) By a simple coding trick, the empty set can be represented as well.

On the other hand it seems striking that all (counter-)examples of non- computability make use of lower dimensional sets like points or lines in 2D or surfaces in 3D. This turns out to be no coincidence and we will therefore, in the present work, restrict to the hyperspace of full-dimensional sets:

Definition 3.1. A subset $R \subseteq \mathbb{R}^d$ is regular iff it coincides with the topological closure of its interior: $R = \overline{R^\circ}$. □

In Engineering Sciences, regular sets are considered an appropriate model for real physical objects in that they reflect the property of being 'solid' [6]. Regularity is a purely topological notion [15]. For the Euclidean case, it generalizes the full-dimensionality condition from convex sets:

Lemma 3.2. a) *A non-empty closed* convex *set* $R \subseteq \mathbb{R}^d$ *is regular iff it contains an interior point:* $R^\circ \neq \emptyset$.

b) *A set* $R \subseteq \mathbb{R}^d$ *is regular iff* $R = \overline{U}$ *for* some *open set* $U \subseteq \mathbb{R}^d$; *equivalently: iff* $R = \overline{A^\circ}$ *for* some *closed set* $A \subseteq \mathbb{R}^d$; *cf. Figure 2.*

c) *The class of regular subsets is closed under these operations: finite unions and topological closure of arbitrary unions.*

$$R_i = \overline{U_i} \quad \forall i \in I \qquad \Longrightarrow \qquad \overline{\bigcup_{i \in I} R_i} = \overline{U}, \qquad U := \bigcup_{i \in I} U_i$$

d) *A closed set* $R \subseteq \mathbb{R}^d$ *is regular iff* $d_R = d_{R^\circ}$.

e) *For open sets* U *and closed sets* A, *respectively,* $\overline{\overset{\circ}{\overline{U}}} = \overline{U}$ *and* $\overset{\circ}{\overline{\overset{\circ}{A}}} = \overset{\circ}{A}$.

f) *The mapping* $A \mapsto \overline{A^\circ}$ *on closed sets (called* regularization*) is neither* $(\psi^d_< \sqcap \psi^d_>, \psi^d_<)$*-computable nor* $(\psi^d_< \sqcap \psi^d_>, \psi^d_>)$*-computable.* □

Proof. The proofs of the above claims as well as of the forthcoming lemmata and theorems are deferred to the appendix. □

FIGURE 2: AN OPEN SET U S.T. $\overline{U} = R$ AND A CLOSED SET A S.T. $\overline{A^\circ} = R$.

Definition 3.3. For computations with regular sets in d-space, several encodings are at hand:

- As regular sets are closed, any of the representations $\psi^d_<$ and $\psi^d_>$ can be used;
- Being equivalently characterized by its topological interior A°, one may as well encode this open set using any of $\theta^d_<$, $\theta^d_>$. Let us denote the correspondingly induced representations with $\overset{\circ}{\theta}{}^d_<$ and $\overset{\circ}{\theta}{}^d_>$.

- By virtue of Lemma 3.2b), one might also use these representations to encode any open $U \subseteq \mathbb{R}^d$ satisfying $\overline{U} = R$. The resulting representations will be called $\overline{\theta}^d_<$ and $\overline{\theta}^d_>$, respectively.
- Finally, any closed $A \subseteq \mathbb{R}^d$ satisfying $\overline{A^\circ} = R$ can represent R, cf. Figure 2; this induces $\overline{\psi^{\circ}}^d_<$ and $\overline{\psi^{\circ}}^d_>$. □

Fortunately, we may end this hierarchical construction here: representing $R = \overline{\overline{U}^{\circ}}$ yields nothing new because of Lemma 3.2e). Already the last step might not seem too plausible any more; but as will turn out in Theorem 4.9e), representation $\overline{\psi^{\circ}}^d_>$ is necessary to describe the work [6] within our framework.

A further way of representing a regular set $R \subseteq \mathbb{R}^d$ is to list the rational points of its interior, i.e., the members of $\mathbb{Q}^d \cap R^\circ$; alternatively, report only the *dyadic* rational points $\mathbb{D}^d \cap R^\circ$, that is, those with denominator a power of 2: $\mathbb{D} = \{p/2^k : p \in \mathbb{Z}, k \in \mathbb{N}\} \subset \mathbb{Q}$. Definition 3.4 generalizes this approach from $\mathbb{Q}^d$ or $\mathbb{D}^d$ to arbitrary effectively enumerable dense subsets of $\mathbb{R}^d$.

Definition 3.4. Let $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$ be a computable real vector function, i.e., such that there is some Turing Machine which, upon input of an index $m \in \mathrm{dom}(\nu) \subseteq \mathbb{N}$ and some $k \in \mathbb{N}$, returns a rational vector $q \in \mathbb{Q}^d$ with $\|\nu(m) - q\|_2 \leq 2^{-k}$, cf. [22]. Suppose that $\mathrm{dom}(\nu)$ is recursively enumerable and $\mathrm{range}(\nu)$ is dense. Furthermore assume ν to be injective[3)]. Such ν will be called a dense enumeration in $\mathbb{R}^d$.

A $\vartheta^\nu_<$-name for regular $R \subseteq \mathbb{R}^d$ is a list of all $m \in \mathrm{dom}(\nu)$ satisfying $\nu(m) \in R^\circ$: it enumerates the set $R^\circ \cap Q$, $Q = \mathrm{range}(\nu)$. A $\vartheta^\nu_>$-name for R is a $\vartheta^\nu_<$-name for $\overline{\mathbb{R}^d \setminus R}$, i.e., a list of all $m \in \mathrm{dom}(\nu)$ s.t., $\nu(m) \notin R$. □

The representations $\vartheta^\nu_<$ and $\vartheta^\nu_>$ generalize ϑ_ν considered in Exercise 4.1.11 of [22]. The latter are, so to speak, restricted to half-infinite intervals; they encode $x \in \mathbb{R}$ by a $(\vartheta^\nu_< \sqcap \vartheta^\nu_>)$-name for the regular set $(-\infty, x]$. Therefore it does not come by surprise that some of our results on $\vartheta^\nu_<$ bear close similarity to previously known facts about ϑ_ν; compare e.g. this work's Theorem 4.10g) with Exercise 4.1.11b) in [22].

We will in the following investigate these ten 'basic' representations and their induced notions of computability:

$$
\begin{array}{ccccccc}
+ & : & \theta^{\circ d}_< & \overline{\theta}^d_< & \vartheta^\nu_< & \psi^d_< & \overline{\psi^{\circ}}^d_< \\
- & : & \psi^d_> & \overline{\psi^{\circ}}^d_> & \vartheta^\nu_> & \theta^{\circ d}_> & \overline{\theta}^d_>
\end{array}
$$

Similar to [22], the first five supply positive information about the encoded set whereas the second five do so for negative information.

Let $\eta :\subseteq \mathbb{N} \to \mathbb{Q}^d$ denote some enumeration of the dense set $\mathbb{Q}^d \subseteq \mathbb{R}^d$. We shall furthermore require that η is computable in the classical sense, that is, as a discrete function mapping any integer m to numerators and denominators (again integers) constituting the rational vector $\eta(m)$. This is for example satisfied by the standard numbering given in Definition 3.1.2 of [22].

3) This can be relaxed: it suffices that the set $\{(m, m') \in \mathrm{dom}(\nu)^2 : \nu(m) = \nu(m')\}$ be r.e..

Roughly speaking, a $\vartheta^\eta_<$-name of R allows to effectively find rational points in R°; with a $\psi^d_<$-name, one can compute real points $x \in R$, cf. EXERCISE 5.1.13 in [22]; a $\mathring{\theta}^d_<$-name permits to enumerate open balls in R°. $\overline{\theta}^d_<$ differs from $\mathring{\theta}^d_<$ in that it may make 'errors': the open balls listed need not cover whole R° but possibly miss some nowhere dense subset. Similarly $\overline{\mathring{\psi}}^d_<$, other than $\psi^d_<$, may report also real points *not* belonging to R, but only a nowhere dense set.

The same intuition applies to the five negative representations when substituting R for $\overline{\mathbb{R}^d \setminus R}$. In fact, it holds

Observation 3.5. Let $R \subseteq \mathbb{R}^d$ be regular and $\tilde{R} := \overline{\mathbb{R}^d \setminus R}$. Then,

a) every $\mathring{\theta}^d_<$-name for R is a $\psi^d_>$-name for $\tilde{R}$;
and, vice versa, every $\psi^d_>$-name for R is a $\mathring{\theta}^d_<$-name for $\tilde{R}$.

b) Similarly for $\vartheta^\nu_<$ and $\vartheta^\nu_>$; and vice versa.

c) Similarly for $\overline{\theta}^d_<$ and $\overline{\mathring{\psi}}^d_>$; and vice versa.

d) Similarly for $\psi^d_<$ and $\mathring{\theta}^d_>$; and vice versa.

e) Similarly for $\overline{\mathring{\psi}}^d_<$ and $\overline{\theta}^d_>$; and vice versa. □

4 Results

The present work relates the basic representations to those used in [10, 8, 14, 11, 6, 24]. For ease of reference, we briefly recall their notions:

Definition 4.1 (GRÖTSCHEL/LOVÁSZ/SCHRIJVER'1988). The regular set $R \subseteq \mathbb{R}^d$ is encoded by the weak membership oracle $\omega^d : \mathbb{Q}^d \times \mathbb{Q}_+ \to \{0,1\}$ iff

$$\forall x \in \mathbb{Q}^d \; \forall r \in \mathbb{Q}_+ : \quad \begin{array}{lcr} \omega^d(x,r) = 1 & \Longrightarrow & R \cap \overline{B}(x,r) \neq \emptyset \\ \omega^d(x,r) = 0 & \Longrightarrow & (\mathbb{R}^d \setminus R) \cap \overline{B}(x,r) \neq \emptyset \end{array}$$ □

As the correspondence $\omega^d \mapsto R$ is surjective, this indeed realizes a representation of regular sets in Euclidean d-space. By abuse of names, we will call this representation ω^d, too; and similarly for the subsequent Definitions of this section.

Definition 4.2 (GE/NERODE'1994). The regular set $R \subseteq \mathbb{R}^d$ is Turing located if its distance function d_R according to Equation (1) is computable; any name[2)4)] for this function will serve as a ψ^d-name for R. □

Definition 4.3 (KUMMER/SCHÄFER'1995). A weak membership test for the regular set $R \subseteq \mathbb{R}^d$ is any partial function $\tau^d :\subseteq \mathbb{Q}^d \times \mathbb{Q}_+ \to \{0,1\}$ satisfying

$$\overline{B}(x,r) \subseteq R^\circ \implies \tau^d(x,r) = 1, \qquad \overline{B}(x,r) \subseteq \mathbb{R}^d \setminus R \implies \tau^d(x,r) = 0$$ □

Intended as a technical tool to their proofs, KUMMER and SCHÄFER introduce an interesting variant of the weak membership test, cf. [14, (*iii'*) on p.553]:

Definition 4.4 (KUMMER/SCHÄFER'1995). The modified membership test for the regular set $R \subseteq \mathbb{R}^d$ is the function

$$\mu^d :\subseteq \mathbb{Q}^d \times \mathbb{Q}_+ \ni (x,r) \mapsto \begin{cases} 1 & \text{if } \overline{B}(x,r) \subseteq R^\circ \\ 0 & \text{if } \overline{B}(x,r) \subseteq \mathbb{R}^d \setminus R \\ \bot & \text{else} \end{cases}$$ □

[4)] w.r.t. some fixed admissible representation of continuous real number functions, see [22].

Notice that, in contrast to τ^d, μ^d is *required* to diverge for $B(x,r) \cap \partial R \neq \emptyset$! A similar condition is imposed in the following

Definition 4.5 (Kummer/Schäfer'1995). The weak characteristic function of the regular set $R \subseteq \mathbb{R}^d$ is given by

$$\chi^d :\subseteq \mathbb{Q}^d \ni x \mapsto \begin{cases} 1 & \text{if } x \in R^\circ \\ 0 & \text{if } x \in \mathbb{R}^d \setminus R \\ \bot & \text{else} \end{cases}$$ □

In Definition 3.13 of [11], the author introduces five representations for open sets and proves their uniform equivalence. By restricting to (the interiors of) regular sets, one obtains, for instance, the following

Definition 4.6 (Hertling'1999). Encode the regular set R by its symmetric distance function

$$ds_R \ : \ \mathbb{R}^d \ni x \ \mapsto \begin{cases} -d_{\partial R}(x) & \text{if} \quad x \in R \\ +d_{\partial R}(x) & \text{if} \quad x \notin R \end{cases}$$

Consider any name[2)4)] for this function as a δ^d-name for R. □

Definition 4.7 (Ziegler/Brattka'2001). A regular set $R \subseteq \mathbb{R}^d$ is ξ^d-represented by (the respective TTE-names realizing) effective approximation from below of both distance functions $d_R : \mathbb{R}^d \to \mathbb{R}$ and $d_A : \mathbb{R}^d \to \mathbb{R}$. Here, A denotes some arbitrary closed subset of $\mathbb{R}^d$ such that $R = \overline{\mathbb{R}^d \setminus A}$. □

Definition 4.8 (Edalat/Lieutier'2002). The regular set $R \subseteq \mathbb{R}^d$ is represented by two countable sequences $\big((x_i,r_i),(y_i,t_i)\big)$ of rational centers and radii iff the two open sets $U = \bigcup_i B(x_i,r_i)$ and $V = \bigcup_i B(y_i,t_i)$ satisfy $U \subseteq R \subseteq \mathbb{R}^d \setminus V$ and $\overline{U} \cup \overline{V} = \mathbb{R}^d$. Such sequences will then be called a π^d-name for R. □

Recall that η denotes some[5)] standard numbering of $\mathbb{Q}^d$. In our first result, we prove that each of the above representations is uniformly equivalent to the join of two basic representations, one encoding positive and one encoding negative information:

Theorem 4.9.

a) $\omega^d \equiv \psi^d_< \sqcap \theta^{\circ d}_>$

b) $\tau^d \ \equiv \ \psi^d_< \sqcap \theta^{\circ d}_>$

c) $\mu^d \equiv \theta^{\circ d}_< \sqcap \psi^d_>$

d) $\chi^d \equiv \vartheta^\eta_< \sqcap \vartheta^\eta_>$

e) $\pi^d \equiv \overline{\theta}^d_< \sqcap \overline{\psi}^{\circ d}_>$

f) $\xi^d \equiv \overline{\theta}^d_< \sqcap \psi^d_>$

g) $\psi^d \equiv \psi^d_< \sqcap \psi^d_>$

h) $\delta^d \equiv \theta^{\circ d}_< \sqcap \psi^d_>$ □

In words: Computability of the weak membership oracle of a regular set R is uniformly equivalent to approximating *two* distance functions from *above*: that of R and the one of R's complement (closed or not, cf. Lemma 3.2d).

The modified membership test for a regular set is uniformly equivalent to approximating from *below* the distance functions of this set and of its complement; it is also uniformly equivalent to δ^d and hence to the other representations introduced in Definition 3.13 of [11], too.

Item g) has in fact already been shown in [3]; it is included here for completeness.

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5) Theorem 4.10g) will reveal that the exact choice does not matter, anyway.

4.1 Basic Representations

We now compare in a systematic way the ten basic representations for regular sets with one another:

T h e o r e m 4.10. *Fix some dense enumeration* $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$*.*

$$\text{a)}\quad \mathring{\theta}^d_< \;\preceq\; \begin{array}{ccc} \preceq & \overline{\theta}^d_< & \preceq \\ \preceq & \vartheta^\nu_< & \preceq \end{array} \;\preceq\; \psi^d_< \;\preceq\; \overline{\mathring{\psi}}^d_<$$

$$\text{b)}\quad \psi^d_> \;\preceq\; \begin{array}{ccc} \preceq & \overline{\mathring{\psi}}^d_> & \preceq \\ \preceq & \vartheta^\nu_> & \preceq \end{array} \;\preceq\; \mathring{\theta}^d_> \;\preceq\; \overline{\theta}^d_>$$

This means that, among the five positive (negative) representations, conversion is possible from left to right in a uniform way. Converting names in reverse direction is however uncomputable even when supplying in addition strongest information of opposite sign:

$$\text{c)}\quad \mathring{\theta}^d_< \sqcap \psi^d_> \;\not\succeq\; \begin{array}{ccc} \not\succeq & \overline{\theta}^d_< \sqcap \psi^d_> & \not\succeq \\ \not\succeq & \vartheta^\nu_< \sqcap \psi^d_> & \not\succeq \end{array} \;\not\succeq\; \psi^d_< \sqcap \psi^d_> \;\not\succeq\; \overline{\mathring{\psi}}^d_< \sqcap \psi^d_>$$

$$\text{d)}\quad \mathring{\theta}^d_< \sqcap \psi^d_> \;\not\succeq\; \begin{array}{ccc} \not\succeq & \mathring{\theta}^d_< \sqcap \overline{\mathring{\psi}}^d_> & \not\succeq \\ \not\succeq & \mathring{\theta}^d_< \sqcap \vartheta^\nu_> & \not\succeq \end{array} \;\not\succeq\; \mathring{\theta}^d_< \sqcap \mathring{\theta}^d_> \;\not\succeq\; \mathring{\theta}^d_< \sqcap \overline{\theta}^d_>$$

The above unconvertability results imply positive and negative representations to be independent: From strongest negative information $\psi^d_>$, for instance, one cannot effectively deduce weak positive information $\psi^d_<$ even when supplying weakest positive $\overline{\mathring{\psi}}^d_<$. Surprisingly, $\overline{\mathring{\psi}}^d_<$ is so weak a representation that it can be deduced from negative information!

e) $\overline{\mathring{\psi}}^d_> \preceq \overline{\mathring{\psi}}^d_<$; *similarly* $\overline{\theta}^d_< \preceq \overline{\theta}^d_>$.

Representations $\vartheta^\nu_<$, $\overline{\theta}^d_<$ are not computably related to one another;
neither are $\vartheta^\nu_>$, $\overline{\mathring{\psi}}^d_>$:

f) $\vartheta^\nu_< \sqcap \psi^d_> \not\preceq \overline{\theta}^d_<$ $\qquad \vartheta^\nu_< \not\succeq \overline{\theta}^d_< \sqcap \psi^d_>$ $\qquad \mathring{\theta}^d_< \sqcap \vartheta^\nu_> \not\preceq \overline{\mathring{\psi}}^d_>$ $\qquad \vartheta^\nu_> \not\succeq \mathring{\theta}^d_< \sqcap \overline{\mathring{\psi}}^d_>$

Each of the two representations enumerating a dense subset forms in fact an entire family on its own. For dense enumerations ν, μ with $Q = \text{range}(\nu)$, $P = \text{range}(\mu)$:

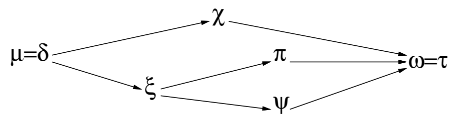
g) $\vartheta^\nu_< \preceq \vartheta^\mu_<$ *iff* $\nu \succcurlyeq \mu$; $\qquad \vartheta^\nu_< \preceq_t \vartheta^\mu_<$ *iff* $Q \supseteq P$. $\qquad$ *Same for* $\vartheta^\nu_>$ *and* $\vartheta^\mu_>$.

We emphasize the similarity to Exercise 4.1.11b in [22]. In TTE, such representations are said to lack *robustness*. □

Relations among the representations given in Definitions 4.1 to 4.8 can now easily be read off from Theorems 4.9 and 4.10. It turns out that μ^d and δ^d are strongest in the sense that all other relevant representations may uniformly be deduced from them; similarly, ω^d and τ^d are equivalent and both weakest.

C o r o l l a r y 4.11. *The representations given in Definitions 4.1 to 4.8 are related to each other as indicated in the directed graph shown below. An arrow from* μ *to* χ

means "$\mu \preccurlyeq \chi$"; absence of an arrow (e.g., between χ and ψ) means computational independence; by transitivity, $\mu \preccurlyeq \omega$. □



4.2 Convex Sets

When restricting to *convex* regular (i.e., full-dimensional) sets, the situation is quite different: most positive basic representations become uniformly equivalent; the negative representations remain distinct but coincide if in addition certain positive information is supplied:

T h e o r e m 4.12. *Let $\mathcal{C}_d$ denote the hyperspace of regular convex sets in $\mathbb{R}^d$ and $\mathcal{B}_d$ the subspace of regular convex sets contained in $[-1,1]^d$; members of $\mathcal{B}_d$ are in particular compact. Fix a dense enumeration ν.*

a) $\mathring{\theta}^d_<\big|^{\mathcal{C}_d} \quad\equiv\quad \overline{\theta}^d_<\big|^{\mathcal{C}_d} \quad\equiv\quad \vartheta^\nu_<\big|^{\mathcal{C}_d} \quad\equiv\quad \psi^d_<\big|^{\mathcal{C}_d}$

The exception is $\overline{\mathring{\psi}}{}^d_<$ which, even restricted to bounded convex sets and when supplying strongest negative information, still remains weaker:

b) $\psi^d_<\big|^{\mathcal{B}_d} \quad\not\preceq\quad \overline{\mathring{\psi}}{}^d_< \sqcap \psi^d_>\big|^{\mathcal{B}_d}$

Also the negative representations remain distinct even restricted to bounded convex regular sets and when supplying weak positive information:

$$\text{c)}\quad \overline{\mathring{\psi}}{}^d_< \sqcap \psi^d_>\big|^{\mathcal{B}_d} \quad\not\preceq\quad \begin{array}{ccc} \not\preceq & \overline{\mathring{\psi}}{}^d_< \sqcap \overline{\mathring{\psi}}{}^d_>\big|^{\mathcal{B}_d} & \not\preceq \\ \not\preceq & \overline{\mathring{\psi}}{}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{B}_d} & \not\preceq \end{array} \quad\not\preceq\quad \overline{\mathring{\psi}}{}^d_< \sqcap \mathring{\theta}^d_>\big|^{\mathcal{B}_d} \quad\not\preceq\quad \overline{\mathring{\psi}}{}^d_< \sqcap \overline{\theta}^d_>\big|^{\mathcal{B}_d}$$

and[6] $\overline{\mathring{\psi}}{}^d_< \sqcap \overline{\mathring{\psi}}{}^d_>\big|^{\mathcal{B}_d} \quad\not\succeq\quad \overline{\mathring{\psi}}{}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{B}_d} \quad\not\succeq\quad \overline{\mathring{\psi}}{}^d_< \sqcap \overline{\mathring{\psi}}{}^d_>\big|^{\mathcal{B}_d}$.

With supply of strong positive information however, they *do* coincide on convex regular sets:

d) $\mathring{\theta}^d_< \sqcap \psi^d_>\big|^{\mathcal{C}_d} \quad\equiv\quad \mathring{\theta}^d_< \sqcap \overline{\mathring{\psi}}{}^d_>\big|^{\mathcal{C}_d} \quad\equiv\quad \mathring{\theta}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{C}_d} \quad\equiv\quad \mathring{\theta}^d_< \sqcap \mathring{\theta}^d_>\big|^{\mathcal{C}_d}$

Again, there is an exception: $\overline{\theta}^d_>$ still remains weaker.

e) $\mathring{\theta}^d_< \sqcap \mathring{\theta}^d_>\big|^{\mathcal{B}_d} \quad\not\preceq\quad \mathring{\theta}^d_< \sqcap \overline{\theta}^d_>\big|^{\mathcal{B}_d}$

Concerning the family of representations enumerating a fixed countable dense set, the positive ones become equivalent for convex sets; the negative ones remain distinct like in Theorem 4.10g) unless strong positive information is provided:

f) *Let ν, μ denote two dense enumerations. Then*

$\vartheta^\nu_<\big|^{\mathcal{C}_d} \equiv \vartheta^\mu_<\big|^{\mathcal{C}_d}, \qquad \overline{\mathring{\psi}}{}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{C}_d} \preccurlyeq \vartheta^\mu_>\big|^{\mathcal{C}_d} \implies \nu \succcurlyeq \mu, \qquad \mathring{\theta}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{C}_d} \equiv \mathring{\theta}^d_< \sqcap \vartheta^\mu_>\big|^{\mathcal{C}_d}$ □

For convex regular sets in $\mathbb{R}^d$, all representations given in Definitions 4.1 to 4.8 are therefore uniformly equivalent:

C o r o l l a r y 4.13.

$\delta^d\big|^{\mathcal{C}_d} \;\equiv\; \mu^d\big|^{\mathcal{C}_d} \;\equiv\; \xi^d\big|^{\mathcal{C}_d} \;\equiv\; \chi^d\big|^{\mathcal{C}_d} \;\equiv\; \pi^d\big|^{\mathcal{C}_d} \;\equiv\; \psi^d\big|^{\mathcal{C}_d} \;\equiv\; \omega^d\big|^{\mathcal{C}_d} \;\equiv\; \tau^d\big|^{\mathcal{C}_d}$ □

[6] The original definition of $\overline{\mathring{\psi}}{}^d_< \sqcap \overline{\mathring{\psi}}{}^d_>$ permits to compose $\psi^d_<$ and $\psi^d_>$-names for *different* closed sets $A_<$ and $A_>$, respectively, provided $\overline{A_<} = \overline{A_>}$. A more restricted variant of this representation might require $A_< = A_>$. Although not equivalent to one another, both versions exhibit, with respect to the remaining representations considered in the present work, the very same computability relations.

This generalizes work of Kummer and Schäfer [14] to the uniform setting, to arbitrary dimension, and to the new representations from Definitions 4.7 and 4.8.

5 Conclusion

We systematically compared, for the case of regular subsets, seven notions of computability suggested in literature. To this end, these notions were related to pairs of 'basic' representations.

For the case of *convex* regular sets, all notions turned out to be equivalent uniformly, in arbitrary dimension, and without requiring boundedness: this generalizes work of Kummer and Schäfer.

For the case of *general* regular sets, the above seven notions exhibit interesting relations shown in Corollary 4.11: The weak membership oracle ω^d of Grötschel, Lovász, and Schrijver, turned out to be uniformly equivalent to Kummer and Schäfer's weak membership test τ^d here, that is, without convexity presumption. On the other hand, these two notions are strictly weaker than, e.g., the ones due to Edalat/Lieutier and Ge/Nerode. Strongest among the seven notions is Kummer and Schäfer's modified membership test μ^d or, equivalently, Hertling's symmetric distance function δ^d. Computability in this sense uniformly implies computability of the other six ones. This reflects that other encodings may, for points x on the boundary of regular set R, behave rather arbitrarily whereas μ^d *must* diverge.

Concerning the basic representations introduced in Definitions 3.3 and 3.4, it is quite a surprise that $\overline{\psi^d_<}$ — although deduced from positive $\psi^d_<$ — is so weak it can be converted from negative $\overline{\psi^d_>}$ (and thus from $\psi^d_>$), cf. Theorem 4.10e). All proofs including the one to this interesting fact can be found in the appendix. Nevertheless we still miss a proof to the following conjecture which says that the *only* negative basic representations which $\overline{\psi^d_<}$ can be converted from are $\psi^d_>$ and $\overline{\psi^d_>}$.

Conjecture 5.1. $\vartheta^\nu_>\big|^{\mathcal{B}_d} \not\preceq \overline{\psi^d_<}\big|^{\mathcal{B}_d}$ □

The representation $\vartheta^\nu_<$ enumerates, from some fixed countable dense subset $Q := \mathrm{range}(\nu)$ of $\mathbb{R}^d$, those points lying in $Q \cap R^\circ$. This bears some similarity to $\psi^d_<$ which can be considered as listing a countable dense subset for R, too [3, 11, 22]. However $\psi^d_<$ may choose the points listed from whole $\mathbb{R}^d$ whereas $\vartheta^\nu_<$ is restricted to those from Q. We therefore investigated its dependence on this subset.

For general regular sets, it is in general not possible to effectively convert from one dense enumeration to another. Such behavior is in TTE referred to as *non-robustness* and considered a disadvantage of such representations. For *convex* regular sets on the other hand, they do become robust. In particular, the Kummer and Schäfer's weak characteristic function on $\mathbb{Q}^d$ may then be substituted, without affecting its computability properties, by the weak characteristic function on the range of *any* dense enumeration.

Let us briefly mention two further representations in this spirit: A $\overline{\vartheta}^\nu_<$-name for regular $R \subseteq \mathbb{R}^d$ is a list of some subset $M \subseteq \mathrm{dom}(\nu)$ such that $\overline{\nu[M]} = R$. Similarly, a $\overline{\vartheta}^\nu_>$-name for R is a $\overline{\vartheta}^\nu_<$-name for $\overline{\mathbb{R}^d \setminus R}$. In contrast to $\vartheta^\nu_<$, $\overline{\vartheta}^\nu_<$ is *not* required to report all $m \in \mathrm{dom}(\nu)$ satisfying $\nu(m) \in R^\circ$ but may omit, e.g., a nowhere dense set.

Correspondingly, it is strictly weaker:

$$\vartheta^\nu_< \preccurlyeq \overline{\vartheta}^\nu_< \quad \text{and} \quad \overline{\vartheta}^\nu_< \not\preccurlyeq \vartheta^\nu_<; \qquad \text{similarly } \vartheta^\nu_> \preccurlyeq \overline{\vartheta}^\nu_> \quad \text{and} \quad \overline{\vartheta}^\nu_> \not\preccurlyeq \vartheta^\nu_> \ .$$

But this weakening has some advantage in terms of slightly increased robustness:

$$(2) \qquad \overline{\vartheta}^\nu_< \preccurlyeq_t \overline{\vartheta}^\mu_< \quad \text{iff} \quad \forall M \subseteq \mathrm{dom}(\nu) : \Big(\overline{\nu[M]} = \overline{\overset{\circ}{\overline{\nu[M]}}} \Longrightarrow \overline{\nu[M]} = \overline{\nu[M] \cap \mathrm{range}(\mu)} \Big)$$

and $\overline{\vartheta}^\nu_< \preccurlyeq \overline{\vartheta}^\mu_<$ iff, in addition to (2), $\nu|^{\mathrm{range}(\mu)} \succcurlyeq \mu$.

The proof to Theorem 4.10e) employs an even more relaxed variant: A $\overline{\vartheta}^{\circ\nu}_<$-name for R is a list of any subset $M \subseteq \mathrm{dom}(\nu)$ such that $R = \overline{\overset{\circ}{\overline{\nu[M]}}}$; similarly for $\overline{\vartheta}^{\circ\nu}_>$. In addition to omitting a discrete subset of R° it may therefore also contain some points $\nu(m)$ *not* lying in R which $\overline{\vartheta}^\nu_<$ must not. Also notice that now, *every* $M \subseteq \mathrm{dom}(\nu)$ is the name of some regular set. The proof to Theorem 4.10e) also shows $\overline{\vartheta}^{\circ\nu}_< \preccurlyeq \overline{\psi}^{\circ d}_<$ (thus, by duality, $\overline{\vartheta}^{\circ\nu}_> \preccurlyeq \overline{\theta}^d_>$). But in fact, the reverse reductions are computable as well, i.e., $\overline{\vartheta}^{\circ\nu}_< \equiv \overline{\psi}^{\circ d}_<$, $\quad \overline{\vartheta}^{\circ\nu}_> \equiv \overline{\theta}^d_>$. This has the surprising consequence that finally these most relaxed variants of ϑ are fully robust, that is, computationally independent of ν. More generally, the following Theorem subsumes our present knowledge about the basic representations from Definitions 3.3 and 3.4 as well as the above relaxations of ϑ.

T h e o r e m 5.2. *Suppose that Conjecture 5.1 holds. Among the 14 basic representations for regular sets considered in the present work, conversions are computable as indicated in Figure 3: uniform reducibility between any two of them is effective if and only*[7] *if a directed path connects the first to the second; in particular* absence *of arrows means*[7] *that conversion is neither computable nor continuous. The latter holds even in a stronger sense: For two positive basic representations, say $\alpha_<$ and $\beta_<$, with no directed path from $\alpha_<$ to $\beta_<$ we have $\alpha_< \sqcap \psi^d_> \not\preccurlyeq_t \beta_<$. In other words: even supply of strongest negative information does not help; similarly, strongest positive information will not permit additional conversions among negative basic representations.* □

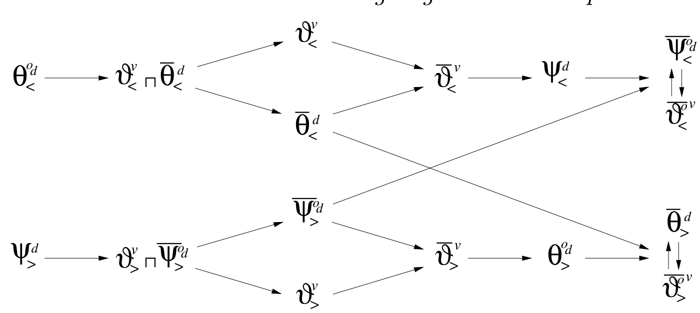


FIGURE 3: COMPUTABILITY RELATIONS AMONG 14 BASIC REPRESENTATIONS.

A final remark: $\overline{\vartheta}^{\circ\nu}_<$ and $\overline{\psi}^{\circ d}_<$ encode a regular set R by a closed set A such that $\overline{A^\circ} = R$, i.e., $A \setminus R$ may be an arbitrary closed nowhere dense subset of $\mathbb{R}^d$ such as (in 3D) lines or surfaces; similarly for $\overline{\vartheta}^{\circ\nu}_>$ and $\overline{\theta}^d_>$. Let $\overline{\vartheta}^{\circ\nu}_<{}'$ and $\overline{\psi}^{\circ d}_<{}'$ denote respective restrictions to those names encoding a closed A for which, in addition to $\overline{A^\circ} = R$, the

[7] provided Conjecture 5.1 turns out to be valid...

set $(A \setminus R) \cap [-n,+n]^d$ is *finite* for each $n \in \mathbb{N}$; similarly for $\overline{\vartheta}^{\circ\nu}_{>}{}'$ and $\overline{\theta}^{d}_{>}{}'$. Our proofs turn out to hold for these representations as well.

6 Future Research

The present work focused on uniform comparison among the above notions of computability. Affirmative results in this sense, such as "$\mathring{\theta}^{d}_{<} \preccurlyeq \vartheta^{\nu}_{<}$", then trivially include *non*-uniform implications like "$\mathring{\theta}^{d}_{<} \Rightarrow \vartheta^{\nu}_{<}$". For negative results, i.e., non-convertability such as "$\vartheta^{\nu}_{<} \not\preccurlyeq \mathring{\theta}^{d}_{<}$", it might be of interest to ask for the non-uniform case: Does it hold, for instance, that $\vartheta^{\nu}_{<} \not\Rightarrow \mathring{\theta}^{d}_{<}$?

Let us also emphasize the remarkable difference between computability of general regular sets and convex ones: For the former, there are many reasonable representations leading to distinct notions as shown in Corollary 4.11. For convex regular sets however, these notions turn out to coincide even uniformly. One might very well wonder what happens in between. We suggest to generalize the class of convex sets to sets of *bounded unconvexity.* The rough idea is that the set R shown in Figure 4 should be assigned unconvexity $1/r$: the larger $r > 0$, the shallower the concave cavity, and the more R resembles a convex set.

More formally, recall the α-hull of $d+1$ points $x_0, \ldots, x_d \in \mathbb{R}^d$ in the sense of [7] as exemplified in Figure 4. Let us call regular set $R \subseteq \mathbb{R}^d$ $1/r$-*unconvex* if, for any $d+1$ points $x_0, \ldots, x_d \in R$, their $(-1/r)$-hull belongs to R. As the 0-hull is but an ordinary simplex, R is convex iff it has 0 (i.e., no) unconvexity. Notice that sets with bounded but nonzero unconvexity in this sense need not even be connected!

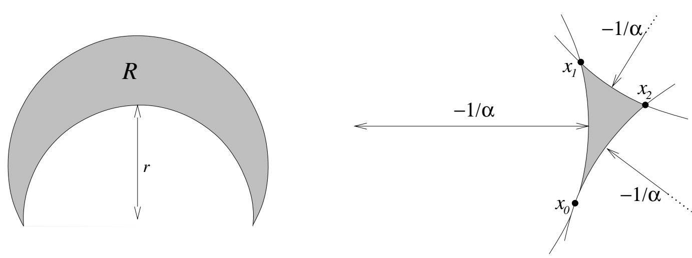


FIGURE 4: A SET WITH UNCONVEXITY $1/r$ AND THE α-HULL OF THREE POINTS IN 2-SPACE.

Review of our proofs for computational equivalence indicates that their two most important concepts — *smallest convex hull* and *core shadow*, cf. Figures 5 and 6 in the appendix — might be extended from simplices to α-hulls. We therefore feel that many results for convex sets may as well hold for sets of bounded unconvexity *provided* a corresponding bound is given.

Returning to general regular sets, we will continue our research and investigate on computability of *operators.* For instance, in [24] it has been shown that intersection $(R_1, R_2) \mapsto R_1 \cap R_2$ of regular subsets is not $(\psi^d \times \psi^d, \psi^d)$-computable but $(\xi^d \times \xi^d, \xi^d)$-computable – *provided* the result $R_1 \cap R_2$ is again a regular set. So how about computability w.r.t. other representations? And how about computability of, say, image $f[\,\cdot\,]$ and pre-image $f^{-1}[\,\cdot\,]$ under suitable classes of functions

$f : \mathbb{R}^d \to \mathbb{R}^d$? Finally, which of the representations are *admissible* [18] in the sense of TTE?

7 Acknowledgment

The author is deeply indebted to PETER HERTLING for loads of support, advice, and seminal feedback. One further merit, it was him to finally find a proof for Conjecture 5.1 which is going to appear in [12]. Theorem 5.2 thus in fact holds unconditionally.

Appendix A (Topology)

Let us, for ease of reference, recall some facts from general topology, cf. [15].

A *topological space* is a set X together with a collection $\mathfrak{O} \supseteq \{\emptyset, X\}$ of subsets of X that is closed under finite intersections and arbitrary unions. Members $U \in \mathfrak{O}$ will be called *open*, complements $A = X \setminus U$ are *closed*. For $M \subseteq X$ let

$$\overline{M} := \bigcap_{\substack{M\subseteq A\subseteq X \\ A \text{ closed}}} A \qquad \text{and} \qquad \overset{\circ}{M} := \bigcup_{\substack{U\subseteq M \\ U \text{ open}}} U$$

denote its *closure* and *interior*, respectively. A *base* for a topological space is a subset $\mathcal{B} \subseteq \mathfrak{O}$ such that each $U \in \mathfrak{O}$ is the union of (arbitrarily many) $B_i \in \mathcal{B}$.
The following facts are straightforward:

i) Closure and interior are *monotone* with respect to set inclusion:

$$M \subseteq N \qquad \Longrightarrow \qquad \overline{M} \subseteq \overline{N}, \quad \overset{\circ}{M} \subseteq \overset{\circ}{N}$$

ii) $\overline{\overline{M}} = \overline{M} \supseteq M, \qquad \big(N^\circ\big)^\circ = N^\circ \subseteq N.$

iii) $\overline{X \setminus M} = X \setminus \overset{\circ}{M}$ and $(X \setminus M)^\circ = X \setminus \overline{M}$.

iv) Suppose U, V are disjoint open sets. Then $\overline{U} \cap V = \emptyset$: this follows from i).

v) In particular if some open ball intersects regular $R = \overline{U}$, it also intersects U.

Recall from Section 2 that, in the framework of TTE [22], computability of arbitrary objects is considered, by means of representations, in terms of computability of their *names*, that is, infinite binary strings $\bar{\sigma} \in \{0,1\}^{\mathbb{N}}$. This set $\{0,1\}^{\mathbb{N}}$ is called *Cantor space* and usually endowed with the *Cantor topology* given by the base $\mathcal{B} = \big\{w\{0,1\}^{\mathbb{N}} : w \in \{0,1\}^*\big\}$. The sometimes so-called *Main Theorem of Computable Analysis* now states that any computable mapping on names $f : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ must necessarily be continuous with respect to the Cantor topology.

In fact, most of our *un*convertability results of type "$\alpha \not\preceq \beta$" rely on this Main Theorem by showing that the mapping on respective α- and β-names induced by a presumed conversion is discontinuous and therefore cannot be computable: "$\alpha \not\preceq_t \beta$".

Appendix B (Proof of Lemma 3.2)

a) See LEMMA 11 in the appendix of [24].

b) If $R = \overline{U}$, then $\overline{R^\circ} = R$ by virtue of e).

c) As $U \subseteq R$, $\overline{U} \subseteq \overline{R} = R$ by monotony. Conversely $U_i \subseteq U$, so $\overline{U_i} \subseteq \overline{U}$ for each i; thus $\bigcup_{i \in I} \overline{U_i} \subseteq \overline{U}$ and, again by A.i), $R = \overline{\bigcup_{i \in I} \overline{U_i}} \subseteq \overline{U}$.

d) Notice that in (1), $d_X = d_Y$ iff $\overline{X} = \overline{Y}$. Hence $d_{R^\circ} = d_{\overline{R^\circ}} = d_R$ for regular R; and conversely $d_{R^\circ} = d_R$ implies $\overline{R^\circ} = R$.

e) $\overset{\circ}{U} \subseteq \overline{U}$, thus $\overline{\overset{\circ}{U}} \subseteq \overline{U}$. $\overline{U} \supseteq U$, hence $\overset{\circ}{\overline{U}} \supseteq U$, therefore $\overline{\overset{\circ}{\overline{U}}} \supseteq \overline{U}$. Similarly for A.

f) Notice that, e.g., $(\psi^d_<, \psi^d_<)$-computing $A \mapsto \overline{A^\circ}$ means nothing but converting $\overline{\psi}^d_<$-names of $R = \overline{A^\circ}$ to respective $\psi^d_<$-names. The claims thus follow from "$\overline{\psi}^d_< \sqcap \psi^d_> \not\preccurlyeq \psi^d_<$" and "$\mathring{\theta}^d_< \sqcap \overline{\psi}^d_> \not\preccurlyeq \psi^d_>$" as proven in Theorem 4.10c) and d).

Appendix C (Proof of Theorem 4.9)

a) and b): We'll show $\quad \psi^d_< \sqcap \mathring{\theta}^d_> \quad \preccurlyeq \quad \omega^d \quad \preccurlyeq \quad \tau^d \quad \preccurlyeq \quad \psi^d_< \sqcap \mathring{\theta}^d_>$

$\omega^d \preccurlyeq \tau^d$: In fact, every weak membership oracle *is* a weak membership test!

For $\overline{B}(x,r) \subseteq R^\circ$, it holds $\omega^d(x,r) = 1$; for otherwise $\omega^d(x,r) = 0$, because ω^d is total, and therefore $\emptyset \neq (\mathbb{R}^d \setminus R) \cap \overline{B}(x,r) \subseteq (\mathbb{R}^d \setminus R^\circ) \cap \overline{B}(x,r)$: a contradiction. Similarly for $\overline{B}(x,r) \subseteq \mathbb{R}^d \setminus R$, it necessarily holds that $\omega^d(x,r) = 0$.

$\psi^d_< \sqcap \mathring{\theta}^d_> \preccurlyeq \omega^d$: It has been shown in [3, 11, 22] that a $\psi^d_<$-name for closed $R \subseteq \mathbb{R}^d$ is uniformly equivalent to a list of (the centers and radii of) all open rational balls intersecting R: By abuse of notation,

$$\psi^d_<(R) \quad \equiv \quad \big\{(y,t) \in \mathbb{Q}^d \times \mathbb{Q}_+ : B(y,t) \cap R \neq \emptyset\big\}$$

Similarly, a $\theta^d_>$-name for R° is (equivalent to) a list of all open rational balls intersecting $\mathbb{R}^d \setminus R^\circ = \overline{\mathbb{R}^d \setminus R}$. The following algorithm will, with the help of these data, realize a weak membership oracle:

Upon input of $(x,r) \in \mathbb{Q}^d \times \mathbb{Q}_+$, scan both lists for an occurrence of (x,r): as $B(x,r)$ intersects either R or $\overline{\mathbb{R}^d \setminus R}$ or both, this search will be successful after finitely many steps. If the (first) occurrence is found in list $\psi^d_<(R)$, report "$\omega^d(x,r) = 1$"; if it is first found in $\theta^d_>(R^\circ)$, answer "$\omega^d(x,r) = 0$".

This indeed yields a valid weak membership oracle: the algorithm terminates for all inputs so that ω^d is total. Furthermore whenever it says "$\omega^d(x,r) = 1$", the assertion $(x,r) \in \psi^d_<(R)$ holds, i.e., $\emptyset \neq B(x,r) \cap R \subseteq \overline{B}(x,r) \cap R$ which justifies this answer. Similarly, "$\omega^d(x,r) = 0$" is reported only in case $B(x,r) \cap (\overline{\mathbb{R}^d \setminus R}) \neq \emptyset$ from which, by virtue of A.v), it follows $\overline{B}(x,r) \cap (\mathbb{R}^d \setminus R) \neq \emptyset$.

$\tau^d \preccurlyeq \psi^d_< \sqcap \mathring{\theta}^d_>$: Fix a weak membership test τ^d for some R. By querying, for different arguments $(x,r) \in \mathbb{Q}^d \times \mathbb{Q}_+$, its values $\tau^d(x,r)$, we will algorithmically generate the lists $\psi^d_<(R)$ and $\theta^d_<(R^\circ)$. Since τ^d may diverge for certain inputs, dove-tailing has to be employed as follows:

Whenever, for some $(x,r) \in \mathbb{Q}^d \times \mathbb{Q}_+$, one finds that $\tau^d(x,r) = 1$, include into the list $\psi^d_<(R)$ all rational balls $B(y,t) \supseteq \overline{B}(x,r)$. This is justified as follows: Since $\tau^d(x,r) = 1$ implies $\overline{B}(x,r) \not\subseteq \mathbb{R}^d \setminus R$, it holds $\emptyset \neq \overline{B}(x,r) \cap R \subseteq B(y,t) \cap R$. Furthermore the thus obtained output is exhaustive: For $B(y,t) \cap R \neq \emptyset$, $B(y,t) \cap R^\circ \neq \emptyset$ because of A.v); hence some open ball $B(x,2r)$ is entirely contained in the non-empty open set $B(y,t)\cap R^\circ$, thus $\tau^d(x,r)=1$ and $B(y,t)\supseteq B(x,2r)\supseteq \overline{B}(x,r)$.

Similarly, whenever $\tau^d(x,r)=0$ holds, report all $B(y,t)\supseteq \overline{B}(x,r)$ in order to eventually obtain the complete list $\theta^d_>(R^\circ)$.

c): We once again refer to [3, 11, 22] where it has been shown that a $\theta^d_<$-name for R° is, up to uniform computational equivalence, a list of *all closed* rational balls contained in R°. Similarly, a $\psi^d_>$-name for R is a list of all closed rational balls contained in $\mathbb{R}^d\setminus R$.

Now the equivalence is obvious: Given such two lists, one may, upon input of $(x,r)\in\mathbb{Q}^d\times\mathbb{Q}_+$, partially compute $\mu^d(x,r)$ by searching these lists; if $(x,r)\in\theta^d_<(R^\circ)$, then report $\mu^d(x,r)=1$; if $(x,r)\in\psi^d_>(R)$, then report $\mu^d(x,r)=0$.

Conversely obtain, from such a partial μ^d, corresponding lists by dove-tailing, for each $(x,r)\in\mathbb{Q}^d\times\mathbb{Q}_+$, computation of $\mu^d(x,r)$: whenever it reports 1, include $\overline{B}(x,r)$ to $\theta^d_<(R^\circ)$; whenever it reports 0, include $\overline{B}(x,r)$ to $\psi^d_>(R)$.

d): Suppose we have oracle access to R's weak characteristic function χ^d. Recall that η was presumed computable as a discrete function with r.e. domain. Dove-tailing thus permits to obtain, by querying the value of $\chi^d\big(\eta^d(m)\big)$ for all $m\in\operatorname{dom}(\eta)$, both sets

$$\Big\{m\in\operatorname{dom}(\eta):\chi^d\big(\eta^d(m)\big)=1\Big\}\quad\text{and}\quad\Big\{m\in\operatorname{dom}(\eta):\chi^d\big(\eta^d(m)\big)=0\Big\}$$

which are respective $\vartheta^\eta_<$- and $\vartheta^\eta_>$-names for R. Conversely upon input of $x\in\mathbb{Q}^d$, one can scan all $m\in\operatorname{dom}(\eta)$ for $\eta(m)=x$ — equality of rational numbers is of course recursive — and then search which of the two lists contains this m (if at all). This yields an effective evaluation of the weak characteristic function $\chi^d(x)$ on $x\in\mathbb{Q}^d\setminus\partial R$ which diverges on $x\in\mathbb{Q}^d\cap\partial R$.

e): Notice that

$$U\subseteq R\subseteq\mathbb{R}^d\setminus V\quad\wedge\quad\overline{U}\cup\overline{V}=\mathbb{R}^d\qquad\Longrightarrow\qquad\overline{U}=R=\mathbb{R}^d\setminus\overset{\circ}{\overline{V}}\ .$$

Indeed $\overline{U}\cup\overline{V}=\mathbb{R}^d$ is equivalent to $\overline{U}\supseteq\mathbb{R}^d\setminus\overline{V}$ which in turn implies $\overline{U}\supseteq\overline{\mathbb{R}^d\setminus\overline{V}}$. Therefore, because of $U\subseteq R\subseteq\mathbb{R}^d\setminus V$, $\quad\overline{U}\subseteq R=\overline{R^\circ}\subseteq\overline{\mathbb{R}^d\setminus\overline{V}}\subseteq\overline{U}$.

Conversely, open U and closed A such that $\overline{U}=R=\overline{A^\circ}$ satisfy $\overline{U}\cup\overline{\mathbb{R}^d\setminus A}=\mathbb{R}^d$.

Now, again by virtue of [3, 11, 22], the two countable sequences $\big(B(x_i,r_i)\big)_{i\in\mathbb{N}}$ and $\big(B(y_i,t_i)\big)_{i\in\mathbb{N}}$ of rational balls are uniformly equivalent to a $\theta^d_<$-name for U and a $\psi^d_>$-name for $A:=\mathbb{R}^d\setminus V$.

Because of $\overline{U}=R$, this $\theta^d_<$-name for U is in fact a $\overline{\theta}^d_<$-name for R; and because of $\overline{A^\circ}=\overline{\mathbb{R}^d\setminus\overline{V}}=R$, the $\psi^d_>$-name for A is at the same time a $\overline{\psi}^d_>$-name for R, too. And vice versa.

f): holds by definition, cf. [24];

g): was already shown in [3, 11, 22].

h): By virtue of Theorem 3.14 in [11], a δ^d-name for R is uniformly equivalent to the join of

i) a $\theta^d_<$-name for R°,

ii) a $\theta^d_>$-name for R°,

iii) a $\theta^d_<$-name for $\mathbb{R}^d \setminus \overline{R^\circ} = \mathbb{R}^d \setminus R$,
iv) a $\theta^d_>$-name for $\mathbb{R}^d \setminus \overline{R^\circ} = \mathbb{R}^d \setminus R$.

Because of Observation 3.5, the latter two are but respective $\psi^d_>$- and $\psi^d_<$-names of R. Hence by merely dropping parts ii) and iv) of the information supplied, one obtains a valid $\mathring{\theta}^d_< \sqcap \psi^d_>$-name for R.

Conversely, these parts can be reconstructed from the remaining i) and iii): the mapping $R^\circ \mapsto \overline{R^\circ} = R$ from open to closed sets is $(\theta^d_<, \psi^d_<)$-computable [11, Lemma 3.9]; and similarly, the mapping $R \mapsto R^\circ$ from closed to open sets is $(\psi^d_>, \theta^d_>)$-computable.

Appendix D (Proof of Theorem 4.10)

a) $\boldsymbol{\mathring{\theta}^d_< \preccurlyeq \vartheta^\nu_<}$**:** As $\mathrm{dom}(\nu)$ is presumed r.e. and $\nu : \mathrm{dom}(\nu) \to \mathbb{R}^d$ computable, one can effectively list all $m \in \mathrm{dom}(\nu)$ for which $\nu(m)$ lies in some given open rational cube $\prod_{i=1}^d (a_i, b_i)$; in fact, this holds even uniformly in $a_1, b_1, \ldots, a_d, b_d \in \mathbb{Q}$. In [3, 11, 22] it has been shown that a $\theta^d_<$-name for open $U \subseteq \mathbb{R}^d$ can uniformly be converted to a list of open rational cubes covering exactly U. We thus obtain an effective enumeration of all $m \in \mathrm{dom}(\nu)$ satisfying $\nu(m) \in U = R^\circ$.

$\boldsymbol{\mathring{\theta}^d_< \preccurlyeq \overline{\theta}^d_<}$**:** As the set $U := R^\circ$ satisfies $\overline{U} = R$, every $\mathring{\theta}^d_<$-name for R is a $\overline{\theta}^d_<$-name as well.

$\boldsymbol{\overline{\theta}^d_< \preccurlyeq \psi^d_<}$**:** We have to, upon input of a $\theta^d_<$-name for open U, output a $\psi^d_<$-name for $R = \overline{U}$. It has been shown in [11] that topological closure is a computable operation in this sense.

$\boldsymbol{\vartheta^\nu_< \preccurlyeq \psi^d_<}$**:** As $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$ is by presumption a computable real vector function one obtains, by restricting ν to the list M of $m \in \mathrm{dom}(\nu)$ with $\overline{\nu[M]} = R$, by virtue of Lemma 5.1.10 in [22] a $\psi^d_<$-name for R.

$\boldsymbol{\psi^d_< \preccurlyeq \overline{\psi}^{\circ d}_<}$**:** is trivial because, by definition, every $\psi^d_<$-name for R can serve as a $\overline{\psi}^{\circ d}_<$-name for R.

b) This follows from a) by virtue of Observation 3.5.

c) $\boldsymbol{\overline{\psi}^{\circ d}_< \sqcap \psi^d_> \not\preccurlyeq_t \psi^d_<}$**:** This follows from Theorem 4.12b).

$\boldsymbol{\psi^d_< \sqcap \psi^d_> \not\preccurlyeq_t \overline{\theta}^d_<}$**:** This follows from $\vartheta^\nu_< \sqcap \psi^d_> \not\preccurlyeq_t \overline{\theta}^d_< \sqcap \psi^d_>$ (f) and $\vartheta^\nu_< \sqcap \psi^d_> \preccurlyeq \psi^d_< \sqcap \psi^d_>$ (a).

$\boldsymbol{\psi^d_< \sqcap \psi^d_> \not\preccurlyeq_t \vartheta^\nu_<}$**:** This follows from $\overline{\theta}^d_< \sqcap \psi^d_> \not\preccurlyeq_t \vartheta^\nu_< \sqcap \psi^d_>$ (f) and $\overline{\theta}^d_< \sqcap \psi^d_> \preccurlyeq \psi^d_< \sqcap \psi^d_>$ (a).

$\boldsymbol{\overline{\theta}^d_< \sqcap \psi^d_> \not\preccurlyeq_t \mathring{\theta}^d_<}$**:** This follows similarly from items f) and a).

$\boldsymbol{\vartheta^\nu_< \sqcap \psi^d_> \not\preccurlyeq_t \mathring{\theta}^d_<}$**:** This follows similarly from items f) and a).

d) This follows from c) by virtue of Observation 3.5.

e) $\boldsymbol{\overline{\psi}^{\circ d}_> \preccurlyeq \overline{\psi}^{\circ d}_<}$**:** The proof will proceed in two steps, by first showing $\overline{\psi}^{\circ d}_> \preccurlyeq \overline{\vartheta}^{\circ\nu}_<$ and then $\overline{\vartheta}^{\circ\nu}_< \preccurlyeq \overline{\psi}^{\circ d}_<$. Here, $\overline{\vartheta}^{\circ\nu}_<$ denotes some relaxed variant of the representation $\vartheta^\nu_<$, cf. Section 5. More formally, let $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$ be a dense enumeration.

A $\overline{\vartheta}^{\circ\nu}_<$-name is a list of some $M \subseteq \mathrm{dom}(\nu)$ such that $R = \overline{\mathring{\overline{\nu[M]}}}$.

$\boldsymbol{\overline{\vartheta}^{\circ\nu}_< \preccurlyeq \overline{\psi}^{\circ d}_<}$**:** Let $M \subseteq \mathrm{dom}(\nu)$ be such that $\overline{A^\circ} = R$ for $A := \overline{\nu[M]}$. From M, a $\psi^d_<$-name for A can be obtained the very same way as in the proof of "$\vartheta^\nu_< \preccurlyeq \psi^d_<$". This is obviously a $\overline{\psi}^{\circ d}_<$-name for R.

$\boldsymbol{\overline{\psi}^{\circ d}_> \preccurlyeq \overline{\vartheta}^{\circ\nu}_<}$**:** Input consists of *all closed* rational balls $\overline{B}_n$ contained in $\mathbb{R}^d \setminus A$ for some closed A such that $R = \overline{A^\circ}$.

For any fixed $n \in \mathbb{N}$, it is easy to enumerate all $m \in \mathrm{dom}(\nu)$ such that $\nu(m) \notin \overline{B}_1 \cup \ldots \cup \overline{B}_n$. By restricting to $n = m$, one thus obtains a list of at least all those m satisfying $\nu(m) \in A^\circ$. The output $M \subseteq \mathrm{dom}(\nu)$ therefore has $\overline{\nu[M]} \supseteq \overline{A^\circ} = R$ and by monotony $\overline{\mathring{\overline{\nu[M]}}} \supseteq R$.

But, possibly, M may contain even some more points. Indeed, if $m \in \mathrm{dom}(\nu)$ satisfies $\nu(m) \notin A$ but is covered only by $\overline{B}_n$ with $n > m$, m gets reported anyway. In fact, these additional points may even accumulate; however only locally: they cannot become dense in some open set outside of A.

Formally, suppose $\mathring{\overline{\nu[M]}} \setminus A \neq \emptyset$. Then this non-empty open set contains some open ball; this in turn contains a smaller, non-empty closed rational ball $\overline{B}$:

$$\text{(3)} \qquad \overline{B} \;\subseteq\; \mathring{\overline{\nu[M]}} \setminus A \;\subseteq\; \mathbb{R}^d \setminus A$$

Lying in A's complement, this ball will be among the ones found in the $\psi^d_>$-Name of A, i.e., $\overline{B} = \overline{B}_n$ for some n. So after having reached this position n in the input, the above algorithm will cease to report points $\nu(m) \in \overline{B}$; and before, it has output only finitely many of them. But this finite set cannot be dense in $\overline{B}$, contradicting (3).

The presumption $\mathring{\overline{\nu[M]}} \setminus A \neq \emptyset$ must therefore be wrong, it rather holds

$$\mathring{\overline{\nu[M]}} \;\subseteq\; A \quad\Longrightarrow\quad \mathring{\overline{\nu[M]}} \;\subseteq\; \mathring{A} \quad\Longrightarrow\quad \overline{\mathring{\overline{\nu[M]}}} \;\subseteq\; \overline{\mathring{A}} \;=\; R$$

$\boldsymbol{\overline{\theta}^d_< \preccurlyeq \overline{\theta}^d_>}$**:** now follows from Observation 3.5.

f) $\boldsymbol{\vartheta^\nu_< \sqcap \psi^d_> \not\preccurlyeq_t \overline{\theta}^d_<}$**:** Consider $R = \overline{B}(0,1)$ and some $\vartheta^\nu_< \sqcap \psi^d_>$-name for it. Its $\overline{\theta}^d_<$-name will, at some position, contain an open rational cube C of volume $\delta > 0$. Let $\nu(m_1), \ldots, \nu(m_n)$ be the points listed in the initial segment of R's $\vartheta^\nu_<$-name upon which the initial segment of its $\overline{\theta}^d_<$-name up to occurrence of C depends. This initial $\vartheta^\nu_<$-segment can then serve as initial segment of a $\vartheta^\nu_<$-name for

$$R' \quad := \quad \bigcup_{i=1}^{n} \overline{B}\big(\nu(m_i), \delta'\big)$$

where $\delta' > 0$ may be chosen arbitrarily small such that R' does *not* have volume δ and therefore cannot contain C – a contradiction to the presumed continuity of transformation of $\vartheta^\nu_< \sqcap \psi^d_>$-names to $\overline{\theta}^d_<$-names.

$\boldsymbol{\overline{\theta}^d_< \sqcap \psi^d_> \not\preccurlyeq_t \vartheta^\nu_<}$**:** Fix $x = \nu(m)$ and consider $U(0) := B(x,1) \setminus \{x\}$, $U(\varepsilon) := B(q,1) \setminus \overline{B}(q,\varepsilon)$. $R(\varepsilon) := \overline{U(\varepsilon)}$. It is easy to construct a $\mathring{\theta}^d_<$-name of $U(0)$ — i.e., a $\overline{\theta}^d_<$-name of $R(0) := \overline{U(0)}$ — such that any prefix of it can be

extended to a $\theta^d_<$-name of $U(\varepsilon)$ — i.e., a $\overline{\theta}^d_<$-name of $R(\varepsilon) := \overline{U(\varepsilon)}$ — for some sufficiently small $\varepsilon > 0$.
On the other hand, any $\vartheta^\nu_<$-name for $R(0)$ must, at some position, report x; a corresponding name for $R(\varepsilon)$, $\varepsilon > 0$, may *not* report x. Thus $\vartheta^\nu_<$-names for $R(\cdot)$ cannot depend continuously on $\varepsilon \geq 0$. In particular, conversion from $\overline{\theta}^d_< \sqcap \psi^d_>$ to $\vartheta^\nu_<$ is not computable.

$\boldsymbol{\vartheta^\nu_> \sqcap \mathring{\theta}^d_< \not\preccurlyeq_t \overline{\psi}^d_>}$**:** and

$\boldsymbol{\overline{\psi}^d_> \sqcap \mathring{\theta}^d_< \not\preccurlyeq_t \vartheta^\nu_>}$**:** follow from the above two cases by virtue of Observation 3.5.

g) **range($\boldsymbol{\nu}$) $\supseteq$ range($\boldsymbol{\mu}$) $\Longrightarrow$ $\boldsymbol{\vartheta^\nu_< \preccurlyeq_t \vartheta^\mu_<}$:** For each $n \in \mathrm{dom}(\nu)$ with $\nu(n) \in \mathrm{range}(\mu)$, let $m(n)$ denote the corresponding $m \in \mathrm{dom}(\mu)$ s.t. $\mu(m) = \nu(n)$; $m(n) := \perp$ for $\nu(n) \notin \mathrm{range}(\mu)$. This discrete partial mapping $m :\subseteq \mathrm{dom}(\nu) \to \mathrm{dom}(\mu)$ is trivially continuous although it may not be computable.
Now consider the following mapping from $\vartheta^\nu_<$-names to $\vartheta^\mu_<$-names: To a list $n_1, n_2, \ldots$ of ν-names of points $\nu(n_i) \in R^\circ$, assign the list $m(n_1), m(n_2), \ldots$ of corresponding μ-names but skip all entries $n_i \notin \mathrm{dom}(m)$.
It is easy to check that this mapping on Baire Space $\mathbb{N}^\omega$ is indeed continuous.

$\boldsymbol{\nu \succcurlyeq \mu \implies \vartheta^\nu_< \preccurlyeq \vartheta^\mu_<}$**:** The proof will effectivize the above construction: By presumption one can enumerate all $m \in \mathrm{dom}(\mu)$. For each one, the prerequisite allows to determine a corresponding ν-name n, that is such that $\nu(n) = \mu(m)$. Scan for this n in the input, i.e., the list of all $n \in \mathrm{dom}(\nu)$ satisfying $\nu(n) \in R^\circ$. If n appears, report the m. Use dove-tailing.

$\boldsymbol{\vartheta^\nu_> \preccurlyeq_t \vartheta^\mu_>}$ **$\Longrightarrow$ range($\boldsymbol{\nu}$) $\supseteq$ range($\boldsymbol{\mu}$):** Suppose there exists $x = \mu(m) \notin \mathrm{range}(\nu)$. Consider a continuous conversion function F mapping $\vartheta^\nu_>$-names to $\vartheta^\mu_>$-names. Upon input of $\mathrm{dom}(\nu)$ — i.e., a name for $R = \emptyset$ — it will enumerate $\mathrm{dom}(\mu)$ and in particular m. By continuity, this output depends only on finitely many ν-names $n_1, \ldots, n_N$. Now let $\delta := \min_{i=1}^N \|x - \nu(n_i)\|_2$ and $R' := \overline{B}(x, \delta)$: the initial segment $n_1, \ldots, n_N$ is also the beginning of a valid $\vartheta^\nu_>$-name for R' but m may not appear in any $\vartheta^\mu_>$-name for R': contradiction.

$\boldsymbol{\vartheta^\nu_> \preccurlyeq \vartheta^\mu_> \implies \nu \succcurlyeq \mu}$**:** is implied by Theorem 4.12f).

The rest follows from Observation 3.5.

Appendix E (Proof of Theorem 4.12)

Let us point out that, as the (closed) complement of a convex set is not convex any more, Observation 3.5 cannot be applied here.

a) We show $\boldsymbol{\psi^d_<\big|^{\mathcal{C}_d} \preccurlyeq \mathring{\theta}^d_<\big|^{\mathcal{C}_d}}$. To this end notice that, for any $d+1$ points $x_0, \ldots, x_d \in R^\circ$, the simplex spanned by them lies in R° because of convexity:

$$\mathrm{chull}(x_0, \ldots, x_d) \quad := \quad \left\{ \textstyle\sum_{j=0}^d x_j \lambda_j \;\middle|\; 0 \leq \lambda_j, \sum \lambda_j = 1 \right\} \quad \subseteq \quad R^\circ$$

Our algorithm will therefore, upon input of a list of all open rational cubes $C = \prod_{i=1}^d (a_i, b_i)$ intersecting R, consider all $(d+1)$-tuples of such cubes. Each such cube C_j contains some point $x_j \in R^\circ$, and the corresponding $(d+1)$-tuple spans a polytope $\mathrm{chull}(x_0, \ldots, x_d)$ contained in R°; however we do not know the position of $x_j \in C_j$. This is circumvented by considering the *smallest* polytope P spanned by all possible choices of $x_j \in C_j$. P can be obtained by choosing, from each C_j, $j = 0, \ldots, d$, one corresponding vertex v_j and computing the simplex $\mathrm{chull}(v_0, \ldots, v_d)$: Letting the v_j range over *all* possible choices of vertices — independently 2^d many for each C_j, $j = 0, \ldots, d$ — the intersection of the respective polytopes $\mathrm{chull}(v_0, \ldots, v_d)$ equals this P; see the example in Figure 5.

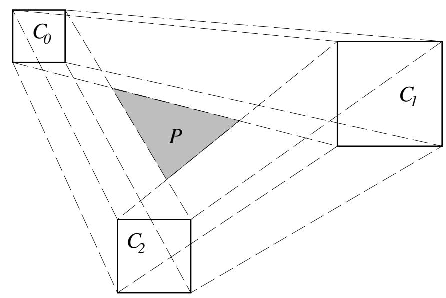


FIGURE 5: THE SMALLEST CONVEX HULL P SPANNED BY THREE RATIONAL CUBES IN 2D.

Notice that P is an open (possibly empty) convex set contained in R°. As each v_j is a vertex of the rational open cube C_j, P can be computed using rational arithmetic only and in finite time. Furthermore it is possible to effectively generate a list of open rational cubes whose union yields P.

Doing this for *all* $(d+1)$-tuples of input cubes C, the thus obtained output of open rational cubes does cover R° because it holds: Every $x \in R^\circ$ belongs to the interior of $\mathrm{chull}(x_0, \ldots, x_d)$ for some $x_0, \ldots, x_d \in \mathbb{Q}^d \cap R^\circ$. This is because open rational simplices are known to form a topological basis of $\mathbb{R}^d$. Then considering sufficiently small cubes containing the points x_i for $i = 0, \ldots, n$ gives the assertion.

b) Consider $R = \overline{B}(0,1) \subseteq [-1,1]^d$ and some $\psi^d_< \sqcap \psi^d_>$-name for it, that is, two respective enumerations of all open rational cubes intersecting R and certain open rational cubes exhausting R's complement. This $\psi^d_< \sqcap \psi^d_>$-name for R can obviously serve as a $\overline{\mathring{\psi}}{}^d_< \sqcap \psi^d_>$-name.

Any conversion to a $\psi^d_<$-name for R will, after finite time, output some open rational cube $C = \prod_{i=1}^d (a_i, b_i)$ intersecting R of size $\max_{i=1}^d |a_i - b_i|$ less than $1/4$. Let $C_{<,1}, \ldots, C_{<,n}$ denote that part of the $\psi^d_<$-input read so far. We will now modify R to R' such that this initial segment is the initial segment of a $\overline{\mathring{\psi}}{}^d_<$-name for R' as well but R' does not intersect C: contradiction.

To this end, notice that there obviously exists some closed rational ball $R' := \overline{B}(x, \frac{1}{4})$ lying in R° that avoids C — simply because R is 'big' and C is 'small'. Choose, from each $C_{<,1}, \ldots, C_{<,n}$, one corresponding point $x_1, \ldots, x_n \in R$ and set $A := R' \cup \{x_1, \ldots, x_n\}$. Then $R' = \overline{A^\circ}$ and $C_{<,1}, \ldots, C_{<,n}$ intersect A; hence the initial segment mentioned above *is* also the initial segment of a $\overline{\mathring{\psi}}{}^d_<$-name for R'.

c) $\boldsymbol{\overline{\mathring{\psi}}{}^d_< \sqcap \overline{\theta}{}^d_> \big|^{\mathcal{B}_d} \not\preceq_t \mathring{\theta}{}^d_> \big|^{\mathcal{B}_d}}$**:** follows from item e).

$\boldsymbol{\overline{\mathring{\psi}}{}^d_< \sqcap \mathring{\theta}{}^d_> \big|^{\mathcal{B}_d} \not\preceq_t \overline{\mathring{\psi}}{}^d_> \big|^{\mathcal{B}_d}}$**:** This time, consider a $\psi^d_< \sqcap \mathring{\theta}{}^d_>$-name for $R = \overline{B}(0, \frac{1}{2})$ which of course is a $\overline{\mathring{\psi}}{}^d_< \sqcap \mathring{\theta}{}^d_>$-name as well. A presumed continuous conversion would, after reading from input only finitely many open rational cubes $C_{<,1}, \ldots, C_{<,n}$ intersecting R and $C_{>,1}, \ldots, C_{>,n}$ intersecting $\mathbb{R}^d \setminus R^\circ$, output some open cube $C \subseteq [-1,1]^d$ not intersecting R.

Let x denote the center of that cube. Now choose points $x_j \in C_{<,j}$ for each $j = 1, \ldots, n$ and choose a radius $\delta > 0$ such that ball $B(x, \delta)$ has volume less than that of the smallest cube $C_{>,j}$. Consider $A := \overline{B}(x, \delta) \cup \{x_1, \ldots, x_n\}$. This set intersects $C_{<,j}$ for each $j = 1, \ldots, n$; it satisfies $\overline{A^\circ} = R'$ for $R' = \overline{B}(x, \delta)$. Furthermore, each $C_{>,j}$ meets R''s complement because R' is too small to cover some $C_{>,j}$. Therefore, the above initial segment of a $\overline{\psi}^d_< \sqcap \mathring{\theta}^d_>$-name for R is an initial segment of a $\overline{\psi}^d_< \sqcap \mathring{\theta}^d_>$-name for R' as well. However, R' *does* intersect the output C (namely in x) of the presumed converting function, a contradiction.

$\boldsymbol{\overline{\psi}^d_< \sqcap \mathring{\theta}^d_>\big|^{\mathcal{B}_d} \not\preceq_t \vartheta^\nu_>\big|^{\mathcal{B}_d}}$**:** is based on the same construction: Re-use the above R and its $\overline{\psi}^d_< \sqcap \mathring{\theta}^d_>$-name; wait for the presumed conversion function to report, after reading $C_{<,1}, \ldots, C_{<,n}$ and $C_{>,1}, \ldots, C_{>,n}$, some point $x = \nu(m)$ not belonging to R. Then consider $A = \overline{B}(x, \delta) \cup \{x_1, \ldots, x_n\}$ to arrive at a contradiction.

$\boldsymbol{\overline{\psi}^d_< \sqcap \overline{\psi}^d_>\big|^{\mathcal{B}_d} \not\preceq_t \psi^d_>\big|^{\mathcal{B}_d}}$**:** follows from

$\boldsymbol{\overline{\psi}^d_< \sqcap \overline{\psi}^d_>\big|^{\mathcal{B}_d} \not\preceq_t \vartheta^\nu_>\big|^{\mathcal{B}_d}}$**:** Let $R := \overline{B}(0, 1/2)$ and $x = \nu(m) \in [-1, 1]^d \setminus R$, $A := R \cup \{x\}$. Fix some $\psi^d_< \sqcap \psi^d_>$-name for A which of course is also a $\overline{\psi}^d_< \sqcap \overline{\psi}^d_>$-name for R. We will here use some different but uniformly equivalent characterization of $\psi^d_>$-names from [3]: A list of *all closed* rational cubes not intersecting A.

The presumed conversion will, after reading open rational cubes $C_{<,1}, \ldots, C_{<,n}$ intersecting A and closed rational cubes $C_{>,1}, \ldots, C_{>,n}$ not intersecting A, report — among others — the fixed element x. Indeed, $x = \nu(m)$ belongs to $[-1, 1]^d \setminus R$ and thus *must* be listed in any $\vartheta^\nu_>$-name for R.

Now choose $x_j \in C_{<,j} \cap A$, $j = 1, \ldots, n$. Also choose $\delta > 0$ smaller than the distance of x to any of the closed cubes $C_{>,j}$: this is possible because $C_{>,j}$ does not intersect A and in particular, as a closed set, has positive distance to $x \in A$.

Consider $A' := \{x_1, \ldots, x_n\} \cup \overline{B}(x, \delta) \cup [-1, +1]^d$. The above initial segment is also that of a $\psi^d_< \sqcap \psi^d_>$-name for A' and thus of a $\overline{\psi}^d_< \sqcap \overline{\psi}^d_>$-name for $R' = \overline{A^\circ} = \overline{B}(x, \delta)$ which *does* contain x: a contradiction to x being reported in R''s $\vartheta^\nu_>$-name.

$\boldsymbol{\overline{\psi}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{B}_d} \not\preceq_t \psi^d_>\big|^{\mathcal{B}_d}}$**:** follows from

$\boldsymbol{\overline{\psi}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{B}_d} \not\preceq_t \overline{\psi}^d_>\big|^{\mathcal{B}_d}}$**:** Consider some $\psi^d_< \sqcap \vartheta^\nu_>$-name for $R = \overline{B}(0, \frac{1}{2})$. Presumed conversion will, after reading open rational cubes $C_{<,1}, \ldots, C_{<,n}$ intersecting R and $\nu(m_1), \ldots, \nu(m_n) \notin R$, report some open rational cube $C \subseteq [-1, +1]^d$ not intersecting R.

Choose, for each $j = 1, ..., n$, some $x_j \in C_{<,j}$ different from $\nu(m_1), ..., \nu(m_n)$. Furthermore take some $c \in C \setminus \big\{\nu(m_1), \ldots, \nu(m_n)\big\}$ and consider $\delta = \min_{j=1}^n |c - \nu(m_j)| > 0$. Letting $A' = \big([-1, +1]^d \cap \overline{B}(c, \delta)\big) \cup \{x_1, \ldots, x_n\}$, the above initial segment can then serve as an initial segment of a $\overline{\psi}^d_< \sqcap \vartheta^\nu_>$-name for $R' = \overline{A^\circ}$, contradicting that C *does* intersect R in c.

d) Our proof for $\boldsymbol{\mathring{\theta}^d_> \sqcap \mathring{\theta}^d_<\big|^{\mathcal{C}_d} \preceq \psi^d_>\big|^{\mathcal{C}_d}}$ is based on the concept of '*core shadows*'.

To this end, recall that a $\mathring{\theta}^d_<$-name of R is computationally equivalent to a list of all closed Euclidean balls $\overline{B}(x,r)$ with rational centers $x \in \mathbb{Q}^d$ and radii $r \in \mathbb{Q}_+$ contained in R°. This is a consequence of representation $\theta^d_<$'s robustness [22]. Similarly, a $\mathring{\theta}^d_>$-name of R° is a list of all open rational balls intersecting $\mathbb{R}^d \setminus R^\circ$.

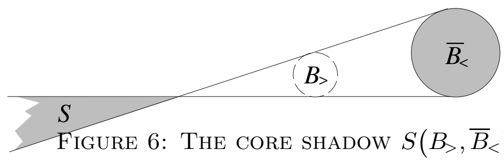


Figure 6: The core shadow $S(B_>, \overline{B}_<)$ cast by two Euclidean balls.

So fix closed $\overline{B}_< \subseteq R^\circ$ and open $B_> \not\subseteq R^\circ$. Their *core shadow*

$$S(B_>, \overline{B}_<) \quad := \quad \Big\{ z \in \mathbb{R}^d \ : \ B_> \subseteq \mathrm{chull}\big(\{z\} \cup \overline{B}_<\big) \Big\}$$

contains exactly those points which, knowing that $\overline{B}_<$ belongs to R° and $B_>$ contains points which don't, can be deduced not to lie in R° because of convexity. Figure 6 depicts an example that illustrates the origin of this notion: an eclipse in astronomy.

From there it is easy to see that, upon input of rational x, r and y, s, one can effectively enumerate open rational balls exhausting the closed (possibly empty, namely iff $r \leq s$) set $S\big(B(y,s), \overline{B}(x,r)\big)$. This enumeration in fact yields a $\psi^d_>$-name for R: it indeed exhausts $\mathbb{R}^d \setminus R$ since the input contains *all* closed rational Euclidean balls $\overline{B}_<$ contained in R° and *all* open rational Euclidean balls intersecting $\mathbb{R}^d \setminus R^\circ$.

e) Let $U := (-\frac{1}{4}, +\frac{1}{4})^d$ be given by some $\theta^d_< \sqcap \theta^d_>$-name, that is, a $\mathring{\theta}^d_< \sqcap \overline{\theta}^d_>$-name for $R = \overline{U}$. Presumed conversion to $\mathring{\theta}^d_>$-names will, after having read open rational cubes $C_{<,1}, \ldots, C_{<,n}$ contained in U and open rational cubes $C_{>,1}, \ldots, C_{>,n}$ intersecting $\mathbb{R}^d \setminus U$, report the open rational cube $C = (-\frac{1}{2}, +\frac{1}{2})^d$. Indeed, a $\mathring{\theta}^d_>$-name for R must list *all* all open rational cubes intersecting $\mathbb{R}^d \setminus R^\circ$.

Now choose for each $j = 1, \ldots, n$ some $x_j \in C_{>,j} \setminus U$. Then consider $U' := [-1,+1]^d \setminus \{x_1, \ldots, x_n\}$. Of the above $\overline{\theta}^d_>$-name for R, the initial part read so far is also the initial segment of a valid $\overline{\theta}^d_>$-name for $R' = \overline{U'}$. Furthermore, $C_{<,1}, \ldots, C_{<,n}$ are contained in $U \subseteq U'$ so that the initial part of the $\mathring{\theta}^d_<$-name for R is valid for R', too. But $C \subseteq \mathring{R}'$ in contradiction to C being reported as intersecting $\mathbb{R}^d \setminus \mathring{R}'$.

f) $\boldsymbol{\vartheta^\nu_<|^{\mathcal{C}_d} \equiv \vartheta^\mu_<|^{\mathcal{C}_d}}$**:** follows from item a).

$\boldsymbol{\overline{\psi}^d_< \sqcap \vartheta^\nu_>|^{\mathcal{C}_d} \preccurlyeq \vartheta^\mu_>|^{\mathcal{C}_d} \implies \nu \succcurlyeq \mu}$**:** In this proof, we will also illustrate how to allow for the generalization indicated in footnote 3). So let $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$ denote a dense enumeration which is not necessarily injective but such that

$$\equiv/\nu \quad := \quad \big\{(m,m') : m,m' \in \mathrm{dom}(\nu), \nu(m) = \nu(m')\big\} \quad \subseteq \quad \mathbb{N}^2$$

is a recursively enumerable set. Furthermore, only compact convex subsets of $\mathbb{R}^d$ (although not necessarily of $[-1,+1]^d$ — this restriction would yield merely $\nu|^{[-1,+1]^d} \succcurlyeq \mu|^{[-1,+1]^d}$) are involved.

As a first step, we strengthen one part of Theorem 4.10g: Fix some continuous conversion from $\overline{\psi}^d_< \sqcap \vartheta^\nu_>$-names of compact convex regular sets R to corresponding $\vartheta^\mu_>$-names. Consider some initial segment of the output (i.e., a finite list $\mu(m_1), \mu(m_2), \ldots, \mu(m_M) \notin R$) depending only (continuity!) on the input's initial part $\nu(n_1), \nu(n_2), \ldots, \nu(n_N) \notin R$ for $\vartheta^\nu_>$ and, for $\overline{\psi}^d_<$, open rational cubes $C_{<,1}, C_{<,2}, \ldots, C_{<,N}$ intersecting A with $\overline{A^\circ} = R$. Then

$$(4) \qquad \{\mu(m_1), \mu(m_2), \ldots, \mu(m_M)\} \quad \subseteq \quad \{\nu(n_1), \nu(n_2), \ldots, \nu(n_N)\}$$

Suppose this is proven. By presumption, some Type-2 Machine M performs the above conversion. We now devise an algorithm that realizes $\mu \preccurlyeq \nu$. Consider input $m \in \text{dom}(\mu)$ for which a corresponding $n \in \text{dom}(\nu)$ is to be found such that $\nu(n) = \mu(m) =: x$. Let $U := B(x+2, 1)$; $R = \overline{U}$ is compact and convex. The mapping $m \mapsto \mu(m) = x$ being computable, it is easy to generate a $\psi^d_<$-name for $A := R$. A list of all $n \in \text{dom}(\nu)$ satisfying $\nu(n) \notin R$ can effectively be output as well because $\text{dom}(\nu)$ is r.e. and $\nu : \text{dom}(\nu) \to \mathbb{R}^d$ is computable.
We thus have a $\overline{\psi}^d_< \sqcap \vartheta^\nu_>$-name for R. Gradually feeding this data into the machine M, it will thus start to generate a $\vartheta^\mu_>$-name for R. As $x = \mu(m) \notin R$, it will after finite time output the m we started with. By virtue of Equation (4), a sought n must be among the input M has read so far; one just has to identify it among these finitely many $n_1, \ldots, n_N$.
It ν is injective, their approximations of ever increasing precision allow to detect and gradually exclude more and more n_i such that $\nu(n_i) \neq \mu(m)$ until in the end exactly one remains: the desired n satisfying $\nu(n) = \mu(n)$. In case ν fails to be injective, there might be several n_i with $\nu(n_i) = \mu(m)$ such that the above exclusion cannot wait for *one* to remain. However, recursively enumerability of $\equiv / \nu$ allows to effectively identify such classes of n_i, i.e., fibers of ν. By now excluding these *classes* until exactly one remains, one still can eventually find a n satisfying $\nu(n) = \mu(n)$.

It remains to prove Equation (4). Suppose on the contrary that $x := \mu(m_M)$ differs from all $\nu(n_i)$, $i = 1, \ldots, N$. As usual, we will construct a different compact convex set $R' = \overline{A'^\circ}$, now *containing* x, such that $\nu(n_i) \notin R'$ and $C_{<,i} \cap A' \neq \emptyset$ for $i = 1, \ldots, N$. The initial segment of the $\overline{\psi}^d_< \sqcap \vartheta^\nu_>$-name for R that $x = \mu(m_M)$ depends on is thus also the initial part of a $\overline{\psi}^d_< \sqcap \vartheta^\nu_>$-name for R'; but m_M may not occur in a $\vartheta^\mu_>$-name for R': contradiction.
To this end choose, for each $i = 1, \ldots, N$, one $x_i \in C_{<,i}$; then let $\delta := \min_{i=1}^N \|x - \nu(n_i)\|_2$ and $A' := \{x_1, \ldots, x_N\} \cup B(x, \delta/2)$.

$\boldsymbol{\mathring{\theta}^d_< \sqcap \vartheta^\nu_>\big|^{\mathcal{C}_d} \equiv \mathring{\theta}^d_< \sqcap \vartheta^\mu_>\big|^{\mathcal{C}_d}}$**:** follows from item d).

Appendix F (Proof of Theorem 5.2)

For showing $\overline{\psi}^d_< \preccurlyeq \overline{\vartheta}^{\circ\nu}_<$, we start with the case of $\nu = \eta :\subseteq \mathbb{N} \to \mathbb{Q}$ being some standard numbering of all rational numbers (1D):

- The algorithm (i.e., a corresponding Type-2 Machine M) receives as input a list of all open rational cubes $C_<$ intersecting A s.t. $R = \overline{A^\circ}$ and successively reports integers $m_i \in \mathrm{dom}(\eta)$ such that eventually $Q = \{\eta(m_i) : i \in \mathbb{N}\}$ satisfies $\overline{\overset{\circ}{Q}} = R$. It proceeds as follows: Upon receipt of $C_<$, test whether any of the points $q_i = \eta(m_i) \in \mathbb{Q}^d$ reported so far lies in this $\overline{C_<}$. If yes, continue; if no, choose and report some new $m \in \mathrm{dom}(\eta)$ such that $\eta(m) \in C_<$.

Output thus contains, to each input $C_<$, at least one point $q = \eta(m) \in \overline{C_<}$: either it already has been reported or a new one is chosen and reported now. As a $\psi^d_<$-name for A contains, for each $a \in A$, cubes $C_< \ni a$ of arbitrarily small width, Q will thus be dense in A: $\overline{Q} \supseteq A$.

$\overline{Q} \setminus A$ may in fact be a non-empty set, however we claim it to contain no interior points. For suppose there exists some open interval $(x, y) = C \subseteq \overline{Q} \setminus A$. Let q denote the first point in C reported by the M; this output was initiated by the receipt of some input $C_< \ni q$ intersecting A in, say, a. Suppose w.l.o.g. that $a > q$; consider the non-empty sub-interval (q, y) of B where Q is by presumption dense, too.

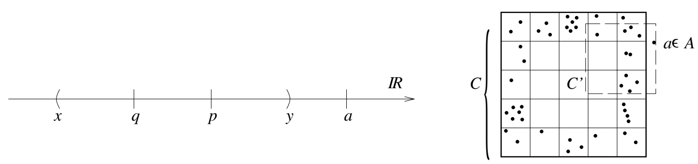


Figure 7: Why the algorithm realizing $\overline{\psi}^d_< \preccurlyeq \overline{\vartheta}^{\circ\nu}_<$ is correct and in particular its output Q is nowhere dense outside A with $\overline{A^\circ} = R$.

After finite time, M will therefore report some further point $p \in (q, y)$. So $C' := (q, p)$ is a non-empty open interval (=cube in $\mathbb{R}^1$) disjoint to A whose vertices q and p have been reported by the above algorithm. Notice that by convexity, *every* cube $C_<$ intersecting both C' and A will necessarily contain one of these vertices as well! Therefore, having reported q and p, M will never again output any point $\eta(m) \in C'$ — contradicting our presumption that Q is dense in whole $C \supseteq C'$.

- The extension of this 1D algorithm to arbitrary dimension d also receives a list of all open rational cubes $C_< = \prod_{i=1}^d (a_i, b_i)$ intersecting A. In fact, we will only consider 'true' cubes with aspect ratio 1, i.e., $|b_i - a_i| = |b_j - a_j|$ for all $1 \le i, j \le d$. Proceeding as above, it is again obvious that the output Q is dense in A. Now suppose $\overline{Q} \setminus A$ contains some open cube $C = \prod_{j=1}^d (c_j - \varepsilon/2, c_j + \varepsilon/2)$ with center $c \in \mathbb{R}^d$ and side length $\varepsilon > 0$. Let C be subdivided into 5^d smaller cubes of side length $\varepsilon/5$ each. We will in particular consider the center one, C', and the $5^d - 3^d$ many boundary subcubes. As Q is by presumption dense in C, M's output will after finite time include points in each of these boundary subcubes. Up to this very instant, C' has received only a finite number of points; and from that on, the above algorithm will never again report points lying in C' for the following reason: Indeed, every input $C_<$ intersecting both C' and A must have side length at least $\frac{2}{5}\varepsilon$ and necessarily covers at least one of the boundary cubes entirely with therein the point(s) already reported. Thus Q cannot be dense in C', contradiction.

- The above algorithm relies in two places on the dense enumeration to be η: for $\mathsf{testing}$ whether none of the points $\eta(m_i)$ reported so far lies in $\overline{C_<}$; and, if this is the case, for $\mathsf{choosing}$ some new $m \in \mathrm{dom}(\eta)$ such that $\eta(m) \in C_<$. Both are easy to perform on rational vectors $\eta(m) \in \mathbb{Q}^d$ but on reals $\nu(m) \in \mathbb{R}^d$ they are in general not recursive. On the other hand, the test is easily seen to be semi-decidable and thus may be replaced by a *dove-tailed* search; a similar property holds for the choice.
- All remaining arrows claimed in Figure 3 are either trivial (e.g., $\vartheta^\nu_< \sqcap \overline{\theta}^d_< \preccurlyeq \vartheta^\nu_<$) or follow by duality (e.g., $\overline{\theta}^d_> \preccurlyeq \overline{\vartheta}^{\circ\nu}_>$) or have already been shown in Theorem 4.10a+b). Absence of further arrows between representations of *different* sign (i.e., positive to negative or negative to positive) is based on Conjecture 5.1. Absence of further arrows *within* one sign class was either already asserted by Theorem 4.10c-f) or is an immediate logical consequence of the latter (e.g., $\vartheta^\nu_< \sqcap \psi^d_> \not\preccurlyeq_t \vartheta^\nu_< \sqcap \overline{\theta}^d_<$ follows from independence of $\vartheta^\nu_< \sqcap \psi^d_>$ and $\overline{\theta}^d_< \sqcap \psi^d_>$, Theorem 4.10f). In fact, only two *non*-reducibility results are missing in order to completely justify Figure 3: $\psi^d_< \sqcap \psi^d_> \not\preccurlyeq_t \overline{\vartheta}^{\circ\nu}_<$ and $\vartheta^\nu_< \sqcap \overline{\theta}^d_< \sqcap \psi^d_> \not\preccurlyeq_t \overset{\circ}{\theta}{}^d_<$. These however can easily be proven by straightforward counter-examples.

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