

Quasi-Optimal Arithmetic for Quaternion Polynomials

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Abstract. Fast algorithms for arithmetic on real or complex polynomials are well-known and have proven to be not only asymptotically efficient but also very practical. Based on *Fast Fourier Transform*, they for instance multiply two polynomials of degree up to n or multi-evaluate one at n points simultaneously within quasi-linear time $\mathcal{O}(n \cdot \text{polylog } n)$. An extension to (and in fact the mere definition of) polynomials over fields \mathbb{R} and \mathbb{C} to the *skew-field* \mathbb{H} of quaternions is promising but still missing. The present work proposes three approaches which in the commutative case coincide but for \mathbb{H} turn out to differ, each one satisfying some desirable properties while lacking others. For each notion, we devise algorithms for according arithmetic; these are quasi-optimal in that their running times match lower complexity bounds up to polylogarithmic factors.

1 Motivation

Nearly 40 years after COOLEY and TUKEY [4], their *Fast Fourier Transform* (FFT) has provided numerous applications, among them

- fast multiplication of polynomials

*Given the coefficients of $p, q \in \mathbb{C}[X]$, $n := \deg(p) + \deg(q)$;
determine the coefficients of $p \cdot q$.*

which, based on FFT, can be performed in $\mathcal{O}(n \cdot \log n)$ and

- their multi-evaluation

*Given the coefficients of $p \in \mathbb{C}[X]$, $\deg(p) < n$, and $x_1, \dots, x_n \in \mathbb{C}$;
determine the values $p(x_1), \dots, p(x_n)$.*

allowing algorithmic solution within $\mathcal{O}(n \cdot \log^2 n)$.

Observe in both cases the significant improvement over naive $\mathcal{O}(n^2)$ approaches. These two examples illustrate a larger class of operations called *Fast Polynomial Arithmetic* [1, 14] with, again, a vast number of applications [7]. For instance, GERASOULIS employed fast polynomial arithmetic to drastically accelerate *N-Body Simulations in 2D* [8], and PAN, REIF, and TATE did so in 3D [11]. Since systems with up to $N = 10^5$ objects arise quite frequently when simulating biochemical processes, the theoretical benefit of asymptotic growth $\mathcal{O}(N \cdot \text{polylog } N)$ over $\mathcal{O}(N^2)$ pays off in practice as well.

Technically speaking in order to calculate, for each of the N particles, the total force it experiences due to the $N - 1$ others, GERASOULIS identifies the plane \mathbb{R}^2 with \mathbb{C} ; he thus turns Coulomb's potential into a rational complex function which, by means of fast polynomial multiplication and multi-evaluation, can be handled efficiently. [11, 13]

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on the other hand exploit fast multi-evaluation of polynomials to *approximate* the total forces in \mathbb{R}^3 . Whether 3D forces can be obtained *exactly* within subquadratic time is still an open question. One promising approach proceeds by identifying, similarly to [8], \mathbb{R}^3 with (a subspace of) HAMILTON's four-dimensional algebra of Quaternions \mathbb{H} and there applying fast polynomial arithmetic of some kind or another. In fact the mere notion of a polynomial becomes ambiguous when passing from fields $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ to the skew-field $\mathbb{K} = \mathbb{H}$. We consider three common approaches to define polynomials (Section 2) and, for each induced kind of quaternion polynomials, present quasi-optimal algorithms supporting according arithmetic operations (Section 3).

2 Quaternions

The algebra \mathbb{H} of quaternions was discovered in 1843 by W.R. HAMILTON in an attempt to extend multiplication of 'vectors' from $\mathbb{R}^2 \cong \mathbb{C}$ to \mathbb{R}^3 . In fact, \mathbb{H} is a *four*-dimensional real vector space whose canonical basis $1, i, j, k$ satisfies the non-commutative multiplicative rule

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k \quad + \text{cyclic interchange} \quad (1)$$

which, by means of associative and distribute laws, is extended to arbitrary quaternions. \mathbb{H} is easily verified to form a *skew-field*, that is, any non-zero element a possesses a unique two-sided multiplicative inverse a^{-1} . In fact it holds $a^{-1} = \bar{a}/|a|^2$ where $\bar{a} := \text{Re}(a) - i \text{Im}_i(a) - j \text{Im}_j(a) - k \text{Im}_k(a)$ is the analogue of complex conjugation and $|a| := \sqrt{a \cdot \bar{a}} = \sqrt{\bar{a} \cdot a} \in \mathbb{R}_+$ the *norm* satisfying $|a \cdot b| = |a| \cdot |b|$. The *center* of \mathbb{H} is \mathbb{R} ; in other words: real numbers and only they multiplicatively commute with any quaternion. For further details, please refer to the excellent¹ CHAPTER 7 of [5]. THEOREM 17.32 in [3] determines the (multiplicative algebraic) complexity of quaternion multiplication; [2] does so similarly for quaternion inversion and division. However rather than on single quaternions, our focus shall lie on asymptotics w.r.t. n , the quaternion polynomials' degree, tending to infinity.

It is well-known that commutativity *has* to be abandoned in order to turn \mathbb{R}^4 into some sort of a field; in fact, FROBENIUS' Theorem states that \mathbb{H} is the *only* associative division algebra beyond $\mathbb{R}^2 \cong \mathbb{C}$. On the other hand to the author's best knowledge, all notions of polynomials either require the ground ring \mathcal{R} to satisfy commutativity or — such as *skew polynomial rings*, see P.262, CHAPTER 16 of [10] — they lack evaluation homomorphisms. The latter means that any polynomial $p = p(X) \in \mathcal{R}[X]$ should naturally induce a mapping $\hat{p} : \mathcal{R} \rightarrow \mathcal{R}, x \mapsto \hat{p}(x)$ such that for all $a, x \in \mathcal{R}$:

$$\hat{X}(x) = x, \quad \hat{a}(x) = a, \quad \widehat{p \cdot q}(x) = \hat{p}(x) \cdot \hat{q}(x), \quad \text{and} \quad \widehat{p + q}(x) = \hat{p}(x) + \hat{q}(x) .$$

The distant goal is to find a notion of quaternion polynomials which naturally generalizes from real or complex ones *and* supports efficient arithmetic by means of, say, quasi-linear time algorithms. Our contribution considers three such definitions for $\mathbb{K}[X]$ which, in case \mathbb{K} is an infinite field, are equivalent to the usual notion. In case $\mathbb{K} = \mathbb{H}$

¹ wrongly condemned in CHAPTER XXI, P.245 of [12]...

however they disagree and give rise to different arithmetic operations. We focus on Multiplication and Multi-Evaluation and present in Section 3, for each of the three notions, according quasi-optimal algorithms.

2.1 Polynomials as Ring of Mappings

The idea pursued in this subsection is that the following objects should be considered polynomials:

- the identity mapping $X := \text{id} : \mathbb{K} \rightarrow \mathbb{K}, x \mapsto x,$
- any constant mapping $\hat{a} : \mathbb{K} \rightarrow \mathbb{K}, x \mapsto a$ for $a \in \mathbb{K}$
- the sum of two polynomials and
- the product of two polynomials.

Formally, let the set $\mathbb{K}^{\mathbb{K}}$ of mappings $f : \mathbb{K} \rightarrow \mathbb{K}$ inherit the ring structure of \mathbb{K} by defining pointwise $f + g : x \mapsto f(x) + g(x), f \cdot g : x \mapsto f(x) \cdot g(x).$ Then embed \mathbb{K} into this ring by identifying $a \in \mathbb{K}$ with the constant mapping $\mathbb{K} \ni x \mapsto a \in \mathbb{K}.$

Definition 1. $\mathbb{K}_1[X]$ is the smallest subring of $\mathbb{K}^{\mathbb{K}}$ containing X and the constant mappings $\mathbb{K}.$ For instance,

$$a_1 + X \cdot a_2 \cdot X \cdot X \cdot a_3 + a_4 \cdot X \cdot X \cdot X \cdot a_5 \in \mathbb{K}_1[X], \quad a_1, \dots, a_5 \in \mathbb{K} \text{ fixed.} \quad (2)$$

$\mathbb{K}_1[X]$ is closed not only under addition and multiplication but also under composition, i.e., $f + g, f \cdot g, f \circ g \in \mathbb{K}_1[X]$ for $f, g \in \mathbb{K}_1[X].$ Since, in the commutative case, any such polynomial can be brought to the form

$$\sum_{\ell=0}^{n-1} a_{\ell} X^{\ell}, \quad n \in \mathbb{N}, \quad a_{\ell} \in \mathbb{K}, \quad (3)$$

Definition 1 there obviously coincides with the classical notion of polynomial rings $\mathbb{R}[X]$ and $\mathbb{C}[X].$ For the skew-field $\mathbb{K} = \mathbb{H}$ of quaternions, the structure of $\mathbb{H}_1[X]$ is not so clear at first sight:

- $a \cdot X \neq X \cdot a$ unless $a \in \mathbb{R}$ i.e., the form (3) in general cannot be attained any more.
- Uniqueness becomes an issue, since

$$X \cdot X \cdot i \cdot X \cdot i + i \cdot X \cdot X \cdot i \cdot X - i \cdot X \cdot i \cdot X \cdot X - X \cdot i \cdot X \cdot X \cdot i \quad (4)$$

vanishes identically [5, TOP OF P.201];

in particular, a polynomial can have many more roots than its 'degree' suggests.

- The fundamental theorem of algebra is violated as well: $i \cdot X - X \cdot i + 1$ has no root in \mathbb{H} [5, P.205].
- Lagrange-style polynomials P_m to pairwise distinct points $x_0, \dots, x_{n-1} \in \mathbb{H},$ e.g.,

$$\left(\prod_{\substack{\ell=0 \\ \ell \neq m}}^{n-1} (X - x_{\ell}) \right) \cdot \left(\prod_{\substack{\ell=0 \\ \ell \neq m}}^{n-1} (x_m - x_{\ell}) \right)^{-1} \quad \text{or} \quad \prod_{\substack{\ell=0 \\ \ell \neq m}}^{n-1} \left((x_m - x_{\ell})^{-1} \cdot (X - x_{\ell}) \right)$$

both interpolate $P_m(x_m) = 1, P_m(x_{\ell}) = 0, m \neq \ell$ but obviously lack uniqueness.

- There is no polynomial division with remainder; e.g. $X \cdot i \cdot X \text{ mod } X^2 = ???$

On the other hand we present in Subsection 3.2 algorithms for addition, multiplication, and multi-evaluation of this kind of quaternion polynomials of degree n in time $\mathcal{O}(n^4 \cdot \text{polylog } n)$. Since it turns out that generic $p \in \mathbb{H}_1[X]$ have roughly n^4 free coefficients, the running time is thus quasi-optimal. Finally, a fast randomized zero-tester for expressions like (2) and (4) comes out easily.

2.2 Polynomials as Sequence of Coefficients

Since the above Definition 1 thus does not allow for quaternion polynomial arithmetic as fast as quasi-linear time, the present subsection proposes another approach. The idea is to identify polynomials with their coefficients. Recall that for $p = \sum_{\ell=0}^{n-1} a_\ell X^\ell$ and $q = \sum_{\ell=0}^{m-1} b_\ell X^\ell$ over a commutative field \mathbb{K} , the finite sequence of coefficients $\mathbf{c} = (c_\ell) \in \mathbb{K}^*$ of $p \cdot q$ is given in terms of $\mathbf{a} = (a_\ell) \in \mathbb{K}^*$ and $\mathbf{b} = (b_\ell) \in \mathbb{K}^*$ by the convolution product

$$\mathbf{c} = \mathbf{a} * \mathbf{b}, \quad c_\ell = \sum_{t=0}^{\ell} a_t \cdot b_{\ell-t}, \quad \ell = 0, \dots, n+m-1 \tag{5}$$

with the implicit agreement that $a_\ell = 0$ for $\ell \geq n$ and $b_\ell = 0$ for $\ell \geq m$.

Definition 2. $\mathbb{K}_2[X]$ is the set \mathbb{K}^* of finite sequences of quaternions, equipped with componentwise addition and convolution product according to (5). Let X denote the special sequence $(0, 1, 0, \dots, 0) \in \mathbb{K}^*$.

It is easy to see that this turns $\mathbb{K}_2[X]$ into a ring which, in case of fields \mathbb{K} of characteristic zero, again coincides with the usual ring of polynomials $\mathbb{K}[X]$. Here the classical results assert that arithmetic operations $+$ and $*$ can be performed within time $\mathcal{O}(n)$ and $\mathcal{O}(n \cdot \log n)$, respectively. In Subsection 3.1, we show that the same is possible in the non-commutative ring $\mathbb{H}_2[X]$. Dealing with n coefficients, this is trivially quasi-optimal.

Unfortunately fast arithmetic for $\mathbb{H}_2[X]$ does not include multi-evaluation, simply because evaluation (substituting X for some $x \in \mathbb{H}$) makes no sense here: One might be tempted to identify $\mathbf{a} \in \mathbb{H}^*$ with the formal expression $\sum_{\ell} a_\ell X^\ell$ and \mathbf{b} with $\sum_{\ell} b_\ell X^\ell$, but then $\mathbf{c} := \mathbf{a} * \mathbf{b}$ does not agree with

$$\left(\sum a_\ell X^\ell \right) \cdot \left(\sum b_\ell X^\ell \right) = \sum_{\ell} \sum_{t=0}^{\ell} a_t \cdot \underbrace{X^t \cdot b_{\ell-t}}_{\neq b_{\ell-t} \cdot X^t} \cdot X^{\ell-t} \neq \sum_{\ell} c_\ell X^\ell$$

because of non-commutativity.

The next subsection considers expressions of the form $\sum a_\ell X^\ell$ as further notion of quaternion polynomials. These lack closure under multiplication; on the other hand, there, multi-evaluation does make sense and turns out to have classical complexity $\mathcal{O}(n \cdot \log^2 n)$.

2.3 One-sided Polynomials

Roughly speaking, one aims at a subclass of $\mathbb{H}_1[X]$ where polynomials have only $\mathcal{O}(n)$ rather than $\Theta(n^4)$ coefficients and thus give a chance for operations with quasi-linear complexity.

Definition 3. Let $X : \mathbb{K} \rightarrow \mathbb{K}$ denote the identity mapping and consider this class of mappings on \mathbb{K} : $\mathbb{K}_3[X] := \{ \sum_{\ell=0}^n a_\ell X^\ell : n \in \mathbb{N}_0, a_\ell \in \mathbb{K} \} \subseteq \mathbb{K}^{\mathbb{K}}$. The degree of $p \in \mathbb{K}_3[X]$ is $\deg(p) = \max_{a_\ell \neq 0} \ell$, $\deg(0) := -1$.

Again this coincides for fields \mathbb{K} of characteristic zero with the usual notions. For the skew-field of quaternions, the restriction compared to (2) applies that all coefficients a_ℓ must be on the left of powers X^ℓ . Unfortunately, this prevents $\mathbb{H}_3[X]$ from being closed under multiplication; fortunately, $\mathbb{H}_3[X]$ has the following other nice properties:

- being a real vector space;
- allows fast multi-evaluation;
- supports interpolation;
- a fundamental theorem of algebra holds;
- polynomials satisfy uniqueness. Formally:

Lemma 4. Consider $p := \sum_{\ell=0}^{n-1} a_\ell X^\ell$, $a_\ell \in \mathbb{H}$.

- a) Suppose $p(x) = 0$ for all $x \in \mathbb{H}$. Then $a_\ell = 0$ for all ℓ .
- b) Nevertheless even $p \neq 0$ may have an infinite (and in particular unbounded in terms of p 's degree) number of roots.
- c) If $a_\ell \neq 0$ for some $\ell \geq 1$, then p has at least one root.

Proof. a) Follows from Lemma 7b) by choosing $n \geq \deg(p)$ and pairwise distinct $x_0, \dots, x_{n-1} \in \mathbb{R}$ since then, no three are automorphically equivalent.
 b) All quaternions $x = i\beta + j\gamma + k\delta$ with $\beta, \gamma, \delta \in \mathbb{R}$ and $\beta^2 + \gamma^2 + \delta^2 = 1$ are easily verified zeros of $p := X^2 + 1$.
 c) Cf. P.205 in [5] or see, e.g., [6].

Interpolation is the question of existence and uniqueness, given x_0, \dots, x_{n-1} and $y_0, \dots, y_{n-1} \in \mathbb{K}$, of a polynomial $p \in \mathbb{K}[X]$ with degree at most $n - 1$ satisfying $p(x_\ell) = y_\ell$ for all $\ell = 0, \dots, n - 1$. In the commutative case, both is asserted for pairwise distinct x_ℓ . Over quaternions, this condition does not suffice neither for uniqueness (Lemma 4b) nor for existence:

Example 5. No $p = aX^2 + bX + c \in \mathbb{H}_3[X]$ satisfies $p(i) = 0 = p(j), p(k) = 1$.

It turns out that here an additional condition has to be imposed which, in the commutative case, holds trivially for distinct x_ℓ , namely being automorphically inequivalent.

Definition 6. Call $a, b \in \mathbb{H}$ automorphically equivalent iff $a = u \cdot b \cdot u^{-1}$ for some non-zero $u \in \mathbb{H}$, that is, iff $\text{Re}(a) = \text{Re}(b) \wedge |\text{Im}(a)| = |\text{Im}(b)|$ where $\text{Im}(a) := i \text{Im}_i(a) + j \text{Im}_j(a) + k \text{Im}_k(a)$.

This obviously is an equivalence relation (reflexivity, symmetry, transitivity). The name comes from the fact that mappings $x \mapsto u \cdot x \cdot u^{-1}$ are exactly the \mathbb{R} -algebra automorphisms of \mathbb{H} ; cf. [5, BOTTOM OF P.215]. The central result of [9] now says:

Lemma 7. For $x_0, \dots, x_{n-1} \in \mathbb{H}$, the following are equivalent

- a) To any $y_0, \dots, y_{n-1} \in \mathbb{H}$, there exists $p \in \mathbb{H}_3[X]$ of $\deg(p) < n$ such that $p(x_\ell) = y_\ell$, $\ell = 0, \dots, n - 1$.
- b) Whenever $p = \sum_{\ell=0}^{n-1} a_\ell X^\ell$ and $q = \sum_{\ell=0}^{n-1} b_\ell X^\ell$ satisfy $p(x_\ell) = q(x_\ell)$ for $\ell = 0, \dots, n - 1$, it follows $a_\ell = b_\ell$.

- c) The Quaternion Vandermonde Matrix $V := (x_\ell^m)_{\ell,m=0,\dots,n-1}$ is invertible.
- d) Its Double Determinant $\|V\|$ does not vanish.
- e) The x_ℓ are pairwise distinct and no three of them are automorphically equivalent.

Concluding this subsection, $\mathbb{H}_3[X]$ has (unfortunately apart from closure under multiplication) several nice structural properties. In 3.3 we will furthermore show that it supports multi-evaluation in time $\mathcal{O}(n \cdot \log^2 n)$. More generally, our algorithm applies to polynomials $\mathbb{H}_3^1[X] := \{ \sum_{\ell=0}^{n-1} a_\ell \cdot X^\ell \cdot b_\ell : n \in \mathbb{N}_0, a_\ell, b_\ell \in \mathbb{H} \}$ with coefficients to both sides of each monomial X^ℓ . This generalized notion has the advantage of yielding not only an \mathbb{R} -vector space but a two-sided \mathbb{H} -vector space.

3 Algorithms

3.1 Convolution of Quaternion Sequences

Beginning with the simplest case of $\mathbb{H}_2[X]$:
 Let $n \in \mathbb{N}$. Given $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{H}^n$ and $\mathbf{b} = (b_0, b_1, \dots, b_{m-1}) \in \mathbb{H}^m$, one can compute their convolution according to (5) from 16 real convolutions² and 12 additions of real sequences within time $\mathcal{O}(n \cdot \log n)$. Indeed write componentwise

$$\mathbf{a} = \text{Re}(\mathbf{a}) + i \text{Im}_i(\mathbf{a}) + j \text{Im}_j(\mathbf{a}) + k \text{Im}_k(\mathbf{a}), \quad \mathbf{b} = \text{Re}(\mathbf{b}) + i \text{Im}_i(\mathbf{b}) + j \text{Im}_j(\mathbf{b}) + k \text{Im}_k(\mathbf{b})$$

and exploit \mathbb{R} -bilinearity of quaternion convolution.

3.2 Ring of Quaternion Mappings

The central point of this subsection is the identification of $\mathbb{H}_1[X]$ with the four-fold Cartesian product of four-variate real polynomials $\prod^4 \mathbb{R}[X_0, X_1, X_2, X_3]$. Formally consider, for $f : \mathbb{H} \rightarrow \mathbb{H}$, the quadruple \tilde{f} of four-variate real functions defined by

$$\begin{aligned} \tilde{f}_0(X_0, \dots, X_3) &:= \text{Re}(p(X_0 + iX_1 + jX_2 + kX_3)) & \tilde{f}_1(X_0, \dots, X_3) &:= \text{Im}_i(p(X_0 + iX_1 + jX_2 + kX_3)) \\ \tilde{f}_2(X_0, \dots, X_3) &:= \text{Im}_j(p(X_0 + iX_1 + jX_2 + kX_3)) & \tilde{f}_3(X_0, \dots, X_3) &:= \text{Im}_k(p(X_0 + iX_1 + jX_2 + kX_3)) \end{aligned} \tag{6}$$

and multiplication among such mappings $\tilde{f}, \tilde{g} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given pointwise by

$$\begin{aligned} (\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \cdot (\tilde{g}_0, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3) &:= \\ (\tilde{f}_0\tilde{g}_0 - \tilde{f}_1\tilde{g}_1 - \tilde{f}_2\tilde{g}_2 - \tilde{f}_3\tilde{g}_3, \tilde{f}_0\tilde{g}_1 + \tilde{f}_1\tilde{g}_0 + \tilde{f}_2\tilde{g}_3 - \tilde{f}_3\tilde{g}_2, \tilde{f}_0\tilde{g}_2 + \tilde{f}_2\tilde{g}_0 + \tilde{f}_3\tilde{g}_1 - \tilde{f}_1\tilde{g}_3, \tilde{f}_0\tilde{g}_3 + \tilde{f}_3\tilde{g}_0 + \tilde{f}_1\tilde{g}_2 - \tilde{f}_2\tilde{g}_1) \end{aligned} \tag{6\frac{1}{2}}$$

In that way, calculations in $\mathbb{H}_1[X]$ can obviously be as well performed in $\prod^4 \mathbb{R}[X_0, \dots, X_3]$. This allows for application of classical algorithms for multivariate polynomials over commutative fields. But before, we need a notion of *degree* on $\mathbb{H}_1[X]$:

Definition 8. For a commutative multi-variate polynomial, let *deg* denotes its total degree; e.g., $\text{deg}(x^2y^3) = 5$, $\text{deg}(0) = -\infty$. The **degree** $\text{deg}(q)$ of a quaternion polynomial $q \in \mathbb{H}_1[X]$ is half the total degree of the real four-variate polynomial $\tilde{f}_0^2 + \dots + \tilde{f}_3^2$ with $\tilde{f}_0, \dots, \tilde{f}_3$ according to (6).

² In fact, 4 complex convolutions suffice; but asymptotically, that gains nothing.

Rather than the total degree, one might as well have considered the maximum one $\deg(x^2y^3) := 3$ since, for 4 variables, they differ by at most a constant factor. However we shall later exploit the equality $\deg(p \cdot q) = \deg(p) + \deg(q)$ valid for the first whereas for the latter in general only the inequality $\deg(p \cdot q) \leq \deg(p) + \deg(q)$ holds. In fact, this nice property carries over to the degree of quaternion polynomials:

Lemma 9. *The degree $\deg(p)$ of $p \in \mathbb{H}_1[X]$ is always integral. Furthermore it holds $\deg(p \cdot q) = \deg(p) + \deg(q)$.*

Now recall the following classical results on four-variate polynomials:

- Lemma 10.** *a) Given (the coefficients of) $p, q \in \mathbb{C}[X_0, \dots, X_3]$, the (coefficients of the) product $p \cdot q$ can be computed in time $\mathcal{O}(n^4 \cdot \log n)$ where $n := \deg(p \cdot q) = \deg(p) + \deg(q)$.*
b) Given $p \in \mathbb{C}[X_0, \dots, X_3]$ of degree n , one can compute within $\mathcal{O}(n^4 \cdot \log n)$ steps the coefficients of $p(T \cdot (X_0, \dots, X_3)^\dagger + \mathbf{y}) \in \mathbb{C}[X_0, \dots, X_3]$, that is, perform on p an affine variable substitution given by $T \in \mathbb{C}^{4 \times 4}$ and $\mathbf{y} \in \mathbb{C}^4$.
c) A given polynomial $p \in \mathbb{C}[X_0, \dots, X_3]$ of degree $n := \deg(p)$ can be evaluated on all n^4 points of a 4-dimensional complex grid $G := A_0 \times A_1 \times A_2 \times A_3$ such that $A_\ell \subseteq \mathbb{C}$, $|A_\ell| = n$, within time $\mathcal{O}(n^4 \cdot \log^2 n)$.
d) The same holds for the regular affine image $G' = T \cdot G + \mathbf{y}$ of such a grid, i.e.,

$$G' = \{T \cdot \mathbf{x} + \mathbf{y} : \mathbf{x} = (x_0, \dots, x_3)^\dagger \in G\}, \quad T \in \mathbb{C}^{4 \times 4} \text{ regular, } \mathbf{y} \in \mathbb{C}^4 .$$

- e) Let $p \in \mathbb{C}[X_0, \dots, X_3]$ be non-zero, $n \geq \deg(p)$. Fix arbitrary $A \subseteq \mathbb{C}$ of size $|A| \geq 2n$. Then, for $(x_0, \dots, x_3) \in A^4$ chosen uniformly at random, the probability of $p(x_0, \dots, x_3) = 0$ is strictly less than $\frac{1}{2}$.*

- Proof.* a) Reduction to the univariate case by means of KRONECKER's embedding: cf. EQUATION (8.3) on p.62 of [1] for $m := 4$; dealing with the complex field \mathbb{C} rather than an arbitrary ring \mathcal{R} of coefficients, the loglog-factor may be omitted.
 b) Folklore. A proof had to be removed from the final version due to space limitations.
 c) Cf. EQUATION (8.5) and the one below on p.63 of [1] for $m := 4$, $c := n$.
 d) follows from b). It is not known whether multi-evaluation is feasible on arbitrarily placed n^4 points within time $\mathcal{O}(n^4 \cdot \text{polylog } n)$.
 e) Cf. SUBSECTION 12.1 in [14].

One could of course identify in a similar way complex univariate polynomials $p \in \mathbb{C}[Z]$ with tuples $p_0, p_1 \in \mathbb{R}[X, Y]$ of real bivariate polynomials. However the thus obtained running times of $\mathcal{O}(n^2 \cdot \text{polylog } n)$ thus obtained for $\mathbb{C}[Z]$ are strikingly suboptimal, basically because *not every* tuple of real bivariate polynomials corresponds to a complex univariate polynomial. For instance, $z \mapsto \text{Re}(z)$ is well-known not only to be no complex polynomial but to even violate RIEMANN-JACOBY's equations of complex differentiability. Surprisingly for quaternion polynomials, the situation is very different:

Lemma 11. $\text{Re}(X) = \frac{1}{4}(X - iXi - jXj - kXk) \in \mathbb{H}_1[X]$. *More generally, every quadruple of real four-variate polynomials corresponds to a quaternion polynomial.*

The generic quaternion polynomial of degree n thus has $\Theta(n^4)$ free coefficients. Lemmas 10 and 11 together yield

- Theorem 12.** a) *Multiplication of two quaternion polynomials $p, q \in \mathbb{H}_1[X]$ is possible in time $\mathcal{O}(n^4 \cdot \log n)$ where $n := \deg(p \cdot q) = \deg(p) + \deg(q)$.*
 b) *Multi-evaluation of p at $x_0, \dots, x_{n-1} \in \mathbb{H}$ can be done within $\mathcal{O}(n^4 \cdot \log^2 n)$, $n := \deg(p)$.*
 c) *Within the same time, multi-evaluation is even feasible at as many as n^4 points x , provided they lie on a (possibly affinely transformed) n^4 -grid G .*

The above complexities are optimal up to the (poly-)logarithmic factor.

Theorem 12 presumes the polynomial(s) to be given as (coefficients of four) real four-variate polynomials. But how fast can one convert input in more practical format like (2) or (4) to that form? By means of fast multiplication of several polynomials, this can be done efficiently as well:

- Theorem 13.** a) *The (ordered!) product $\prod_{\ell=1}^m p_\ell$ of m quaternion polynomials $p_\ell \in \mathbb{H}_1[X]$, each given as quadruple of real four-variate polynomials, can be computed within $\mathcal{O}(n^4 \cdot \log n \cdot \log m)$ where $n = \sum_{\ell} \deg(p_\ell)$ denotes the result's degree.*
 b) *An algebraic expression E over quaternions, i.e., composed from $+$, $-$, \cdot , constants $a \in \mathbb{H}$, and the quaternion variable X — but without powers like X^{99} nor brackets! — can be converted into the quadruple of real four-variate polynomials according to (6) within time $\mathcal{O}(N^4 \cdot \log^2 N)$ where $N = |E|$ denotes the input string's length.*

The above conversion yields a deterministic $\mathcal{O}(N^4 \cdot \log^2 N)$ -test for deciding whether a given quaternion expression like (4) represents the zero polynomial. When satisfied with a randomized test, the same can be achieved much faster:

Theorem 13 (continued)

- c) *Given $\varepsilon > 0$ and an expression E of length $N = |E|$, composed from $+$, $-$, \cdot , constants $a \in \mathbb{H}$, the quaternion variable X , and possibly brackets $(,)$; then one can test with one-sided error probability at most ε whether E represents the zero-polynomial within time $\mathcal{O}(N \cdot \log \frac{1}{\varepsilon})$.*

Proof. a) Standard divide-and-conquer w.r.t. m similar to COROLLARY 2.15 in [3].

- b) Lacking brackets, the input string E necessarily has the form

$$E = E_1 \pm E_2 \pm \dots \pm E_M$$

where E_ℓ describes a product P_ℓ of quaternion constants (degree 0) and the indeterminate X (degree 1). Since obviously $\deg(P_\ell) \leq N_\ell := |E_\ell|$, its real four-variate representation is obtainable within $\mathcal{O}(N_\ell^4 \cdot \log^2 N_\ell)$ steps. Doing so for all $\ell = 1, \dots, M$ leads to running time $\mathcal{O}(N^4 \cdot \log^2 N)$ as $\sum_{\ell} N_\ell \leq N$.

W.l.o.g. let $\deg(P_1) \leq \deg(P_2) \leq \dots \leq \deg(P_M)$. Adding up the just obtained four-variate representations in this increasing order takes additional time $\mathcal{O}(N_1^4 + N_2^4 + \dots + N_M^4) \leq \mathcal{O}(N^4)$.

- c) By virtue of standard amplification it suffices to deal with the case $\varepsilon = \frac{1}{2}$. The algorithm considers any set $A \subseteq \mathbb{R}$ of size $|A| \geq 2N$. It chooses $x_0, x_1, x_2, x_3 \in A$ uniformly and independently at random; and then evaluates the input expression E by substituting $X := x_0 + ix_1 + jx_2 + kx_3$. If the result is zero, the algorithm reports zero, otherwise non-zero.

The running time for evaluation is obviously linear in $|E| = N$. Moreover, only one-sided errors occur. So suppose E represents non-zero $p \in \mathbb{H}_1[X]$. Then obviously $\deg(p) \leq N$ and at least one of the four real four-variate polynomials $\tilde{p}_0, \dots, \tilde{p}_3$ according to (6) is non-zero as well. By virtue of Lemma 10e), this will be witnessed by (x_0, \dots, x_3) — i.e., $p(x_0 + ix_1 + jx_2 + kx_3) \neq 0$ — with probability at least $\frac{1}{2}$. \square

3.3 Multi-Evaluating Two-Sided Polynomials

Consider an expression of the form $p(X) = \sum_{\ell=0}^{n-1} a_\ell X^\ell b_\ell$, $a_\ell, b_\ell \in \mathbb{H}$. Expanding $a_\ell = \text{Re}(a_\ell) + i \text{Im}_i(a_\ell) + j \text{Im}_j(a_\ell) + k \text{Im}_k(a_\ell)$ and similarly for b_ℓ , one obtains, by virtue of distributive laws and since whole \mathbb{R} commutes with X^ℓ , that it suffices to multi-evaluate expressions of the form

$$q(X) = \sum_{\ell=0}^{n-1} \alpha_\ell X^\ell, \quad \alpha_\ell \in \mathbb{R} \quad (!) \tag{7}$$

since $p(X)$ can be obtained from 16 of them, each multiplied both from left and right with some basis element $1, i, j, k$. Now with real α_ℓ , multi-evaluation of (7) is of course trivial on $x_0, \dots, x_{n-1} \in \mathbb{C}$; but we want x_ℓ to be arbitrary quaternions! Fortunately, the latter can efficiently be reduced to the first.

To this end, consider mappings $\varphi_u : \mathbb{H} \rightarrow \mathbb{H}, x \mapsto u \cdot x \cdot u^{-1}$ with $u \in \mathbb{H}$ of norm $|u| = 1$. It is well-known [5, PP.214-216] that, identifying \mathbb{H} with \mathbb{R}^4 , φ_u describes a rotation, i.e., $\varphi_u \in \text{SO}(\mathbb{R}^4)$. Furthermore, restricted to the set

$$\text{Im } \mathbb{H} := \{x \in \mathbb{H} : \text{Re}(x) = 0\} \cong \mathbb{R}^3$$

of *purely imaginary* quaternions, φ_u exhausts whole $\text{SO}(\mathbb{R}^3)$ as u runs through all unit quaternions; this is called **HAMILTON'S THEOREM**. Finally, φ_u is an (and in fact, again, the most general) \mathbb{R} -algebra automorphism, i.e., satisfies for $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{H}$:

$$\varphi_u(\alpha) = \alpha, \quad \varphi_u(x + y) = \varphi_u(x) + \varphi_u(y), \quad \varphi_u(x \cdot y) = \varphi_u(x) \cdot \varphi_u(y) .$$

Lemma 14. *For $v, w \in \text{Im } \mathbb{H}, |v| = 1 = |w|$, let $u := (v+w)/|v+w|$; then $\varphi_u(v) = w$. In particular for $x \in \mathbb{H} \setminus \mathbb{R}, v := \text{Im}(x)/|\text{Im}(x)|, w := i$, it holds $q(x) = u^{-1} \cdot q(y) \cdot u$ where $y := u \cdot x \cdot u^{-1} \in \mathbb{R} + i\mathbb{R} \cong \mathbb{C}$.*

Our algorithm evaluates $q \in \mathbb{R}[X]$ simultaneously at $x_1, \dots, x_n \in \mathbb{H}$ as follows:

- For all $x_\ell \in \mathbb{R} + i\mathbb{R}$, let $u_\ell := 1$;
- for each $x_\ell \notin \mathbb{R}$, compute (in constant time) u_ℓ according to Lemma 14.
- Perform in linear time the transformation $y_\ell := u_\ell \cdot x_\ell \cdot u_\ell^{-1}$.
- Use classical techniques to multi-evaluate q at $y_1, \dots, y_n \in \mathbb{C}$ within $\mathcal{O}(n \cdot \log^2 n)$.
- Re-transform the values $q(y_\ell)$ to $q(x_\ell) = u_\ell^{-1} \cdot q(y_\ell) \cdot u_\ell$.

This proves the claimed running time of $\mathcal{O}(n \cdot \log^2 n)$. \square

4 Conclusion

We proposed three generalizations for the notion 'polynomial' from fields \mathbb{R} and \mathbb{C} to the skew-field \mathbb{H} of quaternions and analyzed their respective properties. For each notion, we then investigated (where applicable) on the algebraic complexity of operations multiplication and multi-evaluation on polynomials in terms of their degree. The upper bounds attained by our respective algorithms match (usually trivial) lower bounds up to polylogarithmic factors.

However since each of the above notions lacks one (e.g., closure under multiplication) or another (e.g., quasi-linear complexity) desirable property, a satisfactory definition for quaternion polynomials is still missing. Here comes another one, generalizing the representation of complex polynomials in terms of their roots:

$$\mathbb{K}_4[X] := \{a_0 \cdot (X - a_1) \cdot (X - a_2) \cdots (X - a_n) : n \in \mathbb{N}_0, a_\ell \in \mathbb{H}\} \quad (8)$$

So what is the complexity for multi-evaluation in $\mathbb{H}_4[X]$?

In view of the planar N -body problem, GERASOULIS' major break-through was fast multi-evaluation of complex rational functions

$$\sum_{\ell=1}^N (X - a_\ell)^{-1} \quad (9)$$

for given $a_1, \dots, a_N \in \mathbb{C}$ at given $x_1, \dots, x_N \in \mathbb{C}$; cf. also COROLLARY 7 in [13]. Our techniques from Subsection 3.3 yield the same for $x_\ell \in \mathbb{H}$ and $a_\ell \in \mathbb{R}$. Thus the crucial question remains whether (9) also allows multi-evaluation in sub-quadratic time for both a_ℓ and x_ℓ being quaternions. But what is a rational quaternion function, anyway? We do not even know what a quaternion polynomial is! Observe that, lacking commutativity,

$$\frac{1}{X - a} + \frac{1}{X - b} = \frac{1}{X - a} \cdot \frac{1}{X - b} \cdot (X - b) + (X - a) \cdot \frac{1}{X - a} \cdot \frac{1}{X - b}$$

cannot be collected into one single fraction, in spite of the common denominator.

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A Postponed Proofs

Here, we collect some proofs which, in the printed version, had to be quelled due to page constraints.

Lemma 15. *Let $f, g \in \mathbb{R}[X_1, \dots, X_d]$ denote d -variate real polynomials. Then,*

$$\deg(f^2 + g^2) = \max\{\deg(f^2), \deg(g^2)\} = 2 \max\{\deg f, \deg g\} .$$

In particular, the total degree of $f^2 + g^2$ is even.

Proof. The second equation is trivial, regarding that the total degree satisfies $\deg(f^2) = 2 \deg(f)$; similarly for the inequality $\deg(f^2 + g^2) \leq \max\{\deg(f^2), \deg(g^2)\}$. One thus has to show that, although in $f^2 + g^2$ certain terms of coinciding maximum total degree might indeed cancel, the above inequality is in fact an equality. This is where the real ground field comes into play: choose in f and g respective terms M and N of coinciding maximum total degree, that is, $M = a \cdot X_1^{m_1} \cdots X_d^{m_d}$ and $N = b \cdot X_1^{n_1} \cdots X_d^{n_d}$ with $\sum m_\ell = \deg f = \deg g = \sum n_\ell$ and $a, b \neq 0$. Then both $M^2 = a^2 \cdot X_1^{2m_1} \cdots X_d^{2m_d}$ and $N^2 = b^2 \cdot X_1^{2n_1} \cdots X_d^{2n_d}$ have total degree equal to $\deg(f^2) = \deg(g^2)$. Furthermore, their respective occurrences in $f^2 + g^2$ cannot cancel because a^2 and b^2 are strictly positive; hence $\deg(f^2 + g^2) \geq \deg(M^2) = \deg(N^2)$. \square

Proof of Lemma 9. Integrality of the degree is covered by applying Lemma 15 above inductively to $(\tilde{f}_0^2, \tilde{f}_1^2)$, $(\tilde{f}_0^2 + \tilde{f}_1^2, \tilde{f}_2^2)$, and $(\tilde{f}_0^2 + \tilde{f}_1^2 + \tilde{f}_2^2, \tilde{f}_3^2)$.

For the second claim, straight-forward calculation confirms the *Four Squares Theorem*³ for real numbers, here applied to the case of real polynomials $\tilde{f}_0, \dots, \tilde{f}_3, \tilde{g}_0, \dots, \tilde{g}_3$:

$$\begin{aligned} & (\tilde{f}_0^2 + \tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_3^2) \cdot (\tilde{g}_0^2 + \tilde{g}_1^2 + \tilde{g}_2^2 + \tilde{g}_3^2) = \\ & = (\tilde{f}_0\tilde{g}_0 - \tilde{f}_1\tilde{g}_1 - \tilde{f}_2\tilde{g}_2 - \tilde{f}_3\tilde{g}_3)^2 + (\tilde{f}_0\tilde{g}_1 + \tilde{f}_1\tilde{g}_0 + \tilde{f}_2\tilde{g}_3 - \tilde{f}_3\tilde{g}_2)^2 \\ & + (\tilde{f}_0\tilde{g}_2 + \tilde{f}_2\tilde{g}_0 + \tilde{f}_3\tilde{g}_1 - \tilde{f}_1\tilde{g}_3)^2 + (\tilde{f}_0\tilde{g}_3 + \tilde{f}_3\tilde{g}_0 + \tilde{f}_1\tilde{g}_2 - \tilde{f}_2\tilde{g}_1)^2 \end{aligned} \quad (10)$$

Now observe that the total degree of the left hand side of (10) is

$$\begin{aligned} & \deg((\tilde{f}_0^2 + \tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_3^2) \cdot ((\tilde{g}_0^2 + \tilde{g}_1^2 + \tilde{g}_2^2 + \tilde{g}_3^2)) \\ & = \deg(\tilde{f}_0^2 + \tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_3^2) + \deg(\tilde{g}_0^2 + \tilde{g}_1^2 + \tilde{g}_2^2 + \tilde{g}_3^2) \end{aligned}$$

which, by Definition 8, agrees with $2 \deg(p) + 2 \deg(q)$ for $p, q \in \mathbb{H}_1[X]$ according to (6). At the same time, in view of Equation (6 $\frac{1}{2}$), the total degree of (10)'s right hand side is nothing but $2 \deg(p \cdot q)$. \square

Proof of Lemma 11.

- Straight forward calculation verifies $\operatorname{Re}(X) = \frac{1}{4}(X - iXi - jXj - kXk)$ which obviously belongs to $\mathbb{H}_1[X]$. Thus, the quadruple of four-variate real polynomials $(X_0, 0, 0, 0) \in \prod^4 \mathbb{R}[X_0, \dots, X_3]$ does correspond to a quaternion polynomial.

³ discovered by EULER in 1748

- Similarly, $(X_1, 0, 0, 0) \in \prod^4 \mathbb{R}[X_0, \dots, X_3]$ corresponds to $\text{Im}_i(X) = \text{Re}(-iX) \in \mathbb{H}_1[X]$; same for $(X_2, 0, 0, 0)$ and $(X_3, 0, 0, 0)$.
- For any real constant α , $(\alpha, 0, 0, 0) \in \prod^4 \mathbb{R}[X_0, \dots, X_3]$ corresponds to $\alpha \in \mathbb{H} \subseteq \mathbb{H}_1[X]$.
- Let $(f, 0, 0, 0)$ and $(\tilde{g}, 0, 0, 0)$ correspond to quaternion polynomials $p, q \in \mathbb{H}_1[X]$, respectively. Then $(f + \tilde{g}, 0, 0, 0)$ corresponds to $p + q \in \mathbb{H}_1[X]$; and,

$$(f \cdot \tilde{g}, 0, 0, 0) \stackrel{(6\frac{1}{2})}{=} (f, 0, 0, 0) \cdot (\tilde{g}, 0, 0, 0)$$

corresponds to $p \cdot q \in \mathbb{H}_1[X]$.

Since $\mathbb{R}[X_0, \dots, X_3]$ is the *smallest* set containing real constants, the generators X_0, X_1, \dots, X_3 , and being closed under addition and multiplication, the above considerations imply that, for any $\tilde{f} \in \mathbb{R}[X_0, \dots, X_3]$, $(\tilde{f}, 0, 0, 0)$ corresponds to some $p \in \mathbb{H}_1[X]$.

- Suppose $(\tilde{f}, 0, 0, 0)$ corresponds to $p \in \mathbb{H}_1[X]$. Then $(0, \tilde{f}, 0, 0)$ corresponds to $-ip \in \mathbb{H}_1[X]$; analogously for $(0, 0, \tilde{f}, 0)$ and $(0, 0, 0, \tilde{f})$.
- Let $(\tilde{f}_0, 0, 0, 0)$ correspond to $p_0 \in \mathbb{H}_1[X]$, $(0, \tilde{f}_1, 0, 0)$ to p_1 , $(0, 0, \tilde{f}_2, 0)$ to p_2 , and $(0, 0, 0, \tilde{f}_3)$ to p_3 . Then $(\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ corresponds to $p_0 + \dots + p_3 \in \mathbb{H}_1[X]$. \square

Proof of Lemma 14. Observe that $u^{-1} = \bar{u}/|u|^2 = -u$ since $u \in \text{Im } \mathbb{H}$ and $|u| = 1$. Thus

$$\varphi_u(v) = -\frac{(v+w)v(v+w)}{|v+w|^2} = -\frac{v^3 + v^2w + wv^2 + wvw}{2 + 2\langle v, w \rangle}$$

because $|v| = 1 = |w|$ by presumption. As $v, w \in \text{Im } \mathbb{H}$, furthermore $v^2 = -1 = w^2$ and thus

$$\varphi_u(v) = -\frac{-v - 2w + (2\langle -v, w \rangle w + v)}{2 + 2\langle v, w \rangle} = w.$$

The \mathbb{R} -algebra homomorphism property ensures that $q(\varphi_u(x)) = \varphi_u(q(x))$ for any polynomial q with real coefficients and $x \in \mathbb{H}$. In particular, \mathbb{R} -linearity yields

$$y = \varphi_u(x) = \varphi_u(\text{Re}(x) + |\text{Im}(x)| \cdot v) = \text{Re}(x) + |\text{Im}(x)|i \in \mathbb{R} + i\mathbb{R}$$

\square