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An explicit solution to Post's Problem over the reals

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Abstract

In the BSS model of real number computations we prove a concrete and explicit semi-decidable language to be undecidable yet not reducible from (and thus strictly easier than) the real Halting Language. This solution to Post's Problem over the reals significantly differs from its classical, discrete variant where advanced diagonalization techniques are only known to yield the *existence* of such intermediate Turing degrees. Then we strengthen the above result and show as well the existence of an uncountable number of incomparable semi-decidable Turing degrees below the real Halting Problem in the BSS model. Again, our proof will give concrete such problems representing these different degrees. Finally we show the corresponding result for the linear BSS model, that is over $(\mathbb{R}, +, -, <)$ rather than $(\mathbb{R}, +, -, \times, \div, <)$.

Keywords: Recursion theory; Real computation

1. Introduction

Is every super-Turing computer capable of solving the discrete Halting Problem H?

More formally, does each undecidable, recursively enumerable language $P \subseteq \mathbb{N}$, when serving as oracle to some appropriate Turing Machine M, enable this M^P to decide H? That question of Post

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[25] was answered to the negative in 1956/1957 independently by Muchnik and Friedberg [13]. [3] Devising the *finite injury priority* sophistication of diagonalization, they proved the existence of r.e. Turing degrees strictly between those of $\emptyset$ and $\emptyset' = H$; cf. [29, Chapters V–VII].

While the diagonal language is also based on a mere existence proof, its reduction to H reveals this as well as many other explicit and practical problems in automatized software verification undecidable.

In contrast, problems like P, though known to exist, so far are quite artificially constructed for the above purpose; see for instance the description of a Turing machine enumerating such a problem in [28, Theorem 1.1]. This is pity as they can have significant impact to the raising field of hypercomputation, that is, (theory) of super-Turing computation. Namely whereas, in spite of e.g. [30], many scientists deny the Halting Problem H to be solvable even by a *non*-Turing device like [16,17], they might be less reluctant toward the solvability of a problem like P because it is strictly easier than H. However, attempts to actually devise a physical system solving P are futile as long as P itself is known no more than to just exist.

It turns out that for real number problems the situation is quite different. More precisely, for the $\mathbb{R}$-machine model due to Blum, Shub, and Smale [5,6], we *explicitly* present a semi-decidable language (specifically, the set $\mathbb{Q}$ of rationals) and prove it to neither be reducible from the real Halting Problem $\mathbb{H}_\mathbb{R}$ nor from the set $\mathbb{A}$ of algebraic reals. The proof exploits that real computability theory, apart from logic as in the discrete case, has also algebraic and topological aspects.

Section 1.2 recalls the basics of real number computation in the BSS model as well as the recursion-theoretic notions of reducibility and degrees; Section 2 contains the first main result of our work; we show $\mathbb{Q} \precneqq \mathbb{A}$, i.e. the real algebraic numbers cannot be decided using a BSS oracle machine which has access to the (undecidable!) set of rationals as oracle set. Section 2.1 proves the "$\preceq$" -part, Section 2.2 the "$\precneqq$" -part. In Section 3 the results are generalized in order to get an uncountable number of incomparable semi-decidable problems below the real Halting Problem. We conclude in Section 5 with some general remarks on hypercomputation.

1.1. Related work

Our contribution adds to other results, indicating that many (separation-) problems which seem to require non-constructive (e.g., diagonalization) techniques in the discrete case, admit an explicit solution over the reals. For instance, a problem neither in $\mathcal{VP}$ nor $\mathcal{VNP}$-complete (provided that $\mathcal{VP} \neq \mathcal{VNP}$, of course) was presented explicitly in [7, §5.5].

Cucker's work [10] is about the Arithmetic Hierarchy over $\mathbb{R}$, that is, degrees beyond the real Halting Problem $\mathbb{H}_\mathbb{R}$.

Hamkins and Lewis considered Post's Problem over the reals for *Infinite Time* Turing Machines, that is, with respect to arguments $x \in \mathbb{R}$ given by their binary expansion and for hypercomputers performing an ordinal number of steps like $1, 2, 3, \ldots, n, \ldots, \omega, \omega + 1, \ldots, 2\omega, \ldots$. They showed in [14] that in this model,

- for sets of reals the answer is "no" just like in the classical discrete case;
- for *single* real numbers x on the other hand, considered as sets $L_x \subseteq \mathbb{N}$ of those indices where the binary expansion of x has a 1, there is no undecidable degree below that of the Halting

[3] The existence of intermediate Turing degrees that need not to be r.e. follows from a result by Kleene and Post in 1954, see [29].

Problem (of Infinite Time Machines). Post's Problem therefore is to be answered to the *positive* in this latter setting!

The existence of different *complexity degrees* below $\mathcal{NP}$ in the BSS model both for real and for complex numbers was studied in [3,9,22] and related to classical results (cf. [19,27]) for the Turing model.

1.2. The BSS model of real number computation

This section summarizes very briefly the main ideas of real number computability theory. For a more detailed presentation, especially a precise definition of BSS machines, see [5,6].

Essentially a (real) BSS machine can be considered as a Random Access Machine over $\mathbb{R}$ which is able to perform the basic arithmetic operations at unit cost and whose registers can hold arbitrary real numbers.

Definition 1 (*Blun et al. [5]*). Let $Y \subseteq \mathbb{R}^\infty := \bigoplus_{k\in\mathbb{N}} \mathbb{R}^k$, i.e. the set of finite sequences of real numbers.

(a) The size of an $x \in \mathbb{R}^k$ is $\text{size}_\mathbb{R}(x) = k$. The cost of any of the operations $\{+, -, *, :\}$ or a test is $x \geqslant 0$? is 1. The cost of an entire computation is the number of operations performed until a machine halts.
(b) A set $A \subseteq \mathbb{R}^\infty$ is called a decision problem or a language over $\mathbb{R}^\infty$. We call a function $f: A \to \mathbb{R}^\infty$ (BSS-) computable iff it is realized by a BSS machine over admissible input set A. Similarly, a set $A \subseteq \mathbb{R}^\infty$ is decidable in $\mathbb{R}^\infty$ iff its characteristic function is computable. It is semi-decidable iff there is a BSS algorithm which takes inputs from $\mathbb{R}^\infty$ and halts precisely on the elements belonging to A.
(c) A BSS oracle machine using an oracle set $B \subseteq \mathbb{R}^\infty$ is a BSS machine with an additional type of node called oracle node. Entering such a node the machine can ask the oracle whether a previously computed element $x \in \mathbb{R}^\infty$ belongs to B. The oracle gives the correct answer at unit cost.

Several further concepts and notions now can be defined straightforwardly. With respect to the following definition note that a BSS machine can be encoded as an element of $\mathbb{R}^\infty$.

Definition 2. The real Halting Problem $\mathbb{H}_\mathbb{R}$ is the following decision problem. Given the code $c_M \in \mathbb{R}^\infty$ of a BSS machine M together with an $x \in \mathbb{R}^\infty$, does M terminate its computation on input x?

Both the existence of such a coding for BSS machines and the undecidability of $\mathbb{H}_\mathbb{R}$ in the BSS model were shown in [5].

Next, oracle reductions are defined as usual.

Definition 3. (a) A real number decision problem A is reducible to another decision problem B if there is a BSS oracle machine that decides membership in A by using B as oracle set. We denote this reducibility by $A \preceq B$. We write $A \precneqq B$ when A is reducible to B, but B is not reducible to A.

(b) If A is reducible to B and vice versa, we write $A \equiv B$. This defines equivalence classes $\{B: A \equiv B\}$ among real number decision problems called (real) Turing degrees or BSS degrees.
(c) If none of two problems is reducible to the other, they are said to be incomparable.

The main question treated in this paper is: Are there incomparable Turing degrees strictly between the degree $\emptyset$ of decidable problems in $\mathbb{R}^\infty$ and the degree $\emptyset'$ of the real Halting Problem $\mathbb{H}_\mathbb{R}$?

2. Explicit solution to Post's Problem over the reals

Consider the sets $\mathbb{Q}$ of all rational numbers and $\mathbb{A}$ of all algebraic reals, that is, of real zeros of polynomials with rational coefficients, only. $\mathbb{Q}$ is obviously semi-decidable (upon input of $x \in \mathbb{R}$, simply check for all pairs of integers $r, s \in \mathbb{Z}$ whether $x = r/s$) but well known not to be decidable [15,23]. In fact the same holds for $\mathbb{A}$: Given $x \in \mathbb{R}$, try for all polynomials $p \in \mathbb{Q}[X]$ whether $p(x) = 0$.

Our first main result states that, even given oracle access to $\mathbb{Q}$, $\mathbb{A}$ remains undecidable: $\mathbb{A} \not\preceq \mathbb{Q}$. Since oracle access to the Halting Problem $\mathbb{H}_\mathbb{R}$ of BSS machines allows to decide $\mathbb{A}$ by querying whether the above search for $p \in \mathbb{Q}[X]$ terminates, $\mathbb{Q}$ thus constitutes an explicit example of a real BSS degree strictly between the decidable one and that of the Halting Problem.

We also show $\mathbb{Q} \preceq \mathbb{A}$.

Theorem 4. *In the BSS model of real number computation it holds* $\mathbb{Q} \precneqq \mathbb{A}$. *In particular, transcendence is not semi-decidable even when using* $\mathbb{Q}$ *as an oracle set.*

This result is, in spite of the notational resemblance to $\mathbb{Q} \subsetneq \mathbb{A}$, by no means obvious.

2.1. *Deciding $\mathbb{Q}$ in $\mathbb{R}$ by means of an $\mathbb{A}$-oracle*

In this section, we prove

Lemma 5. $\mathbb{Q} \preceq \mathbb{A}$.

Proof. Consider some input $x \in \mathbb{R}$. By querying the $\mathbb{A}$-oracle, identify and rule out the case that x is not in $\mathbb{A}$ (and hence not in $\mathbb{Q}$ either). So it remains to distinguish $x \in \mathbb{Q}$ from $x \in \mathbb{A} \setminus \mathbb{Q}$. To this end, calculate $d := \deg(x)$ according to Lemma 6 below and test whether $d = 1$ ($x \in \mathbb{Q}$) or $d \geqslant 2$ ($x \notin \mathbb{Q}$). □

Recall that the *degree* of an algebraic $a \in \mathbb{R}$ is defined to be

$$\deg(a) = \dim_\mathbb{Q} \mathbb{Q}(a) = [\mathbb{Q}(a): \mathbb{Q}],$$

that is, the dimension of the rational extension field generated by a. It is well known, for example in [20, Proposition V, §1.2], that finite field extensions $M \subset K \subset L$ satisfy

$$[L : M] = [L : K] \cdot [K : M]. \tag{1}$$

A non-algebraic number is *transcendental*, the set of which we shall denote by $\mathbb{T}$.

Lemma 6. *The function* $\deg\colon \mathbb{A} \to \mathbb{N},\ a \mapsto \deg(a)$ *is BSS-computable.*

We point out that the restriction of deg to algebraic numbers is essential here; in other words: While for reasons of mathematical convenience one can *define* $\deg(x) := \infty$ for transcendental x, a BSS machine cannot *compute* it.

Proof. Exploit that an alternative yet equivalent definition for $\deg(a)$ is given by the degree of a minimal polynomial of a, that is, of an irreducible $p \in \mathbb{Q}[X]$ of positive degree with $p(a) = 0$ [20, Proposition V, §1.4]. Moreover, p can be chosen from $\mathbb{Z}[X]$ with content (i.e., the gcd of its coefficients equal to) 1. In this case, p is irreducible in $\mathbb{Q}[X]$ iff irreducible in $\mathbb{Z}[X]$: Gauss' Lemma [20, Theorem IV, §2.3].

Therefore we enumerate all non-constant $p \in \mathbb{Z}[X]$ of content 1 and, for each one, plug-in a to test whether $p(a) = 0$. If so, check p for irreducibility—a property in classical $\mathcal{NP}$ by virtue of [8] and thus BSS-decidable. If this test succeeds as well, return $\deg(p)$ and terminate; otherwise continue with the next p. □

Remark 7. An elementary decision procedure for irreducibility in $\mathbb{Z}[X]$ proceeds—although not within nondeterministic polynomial time—as follows:

Given $p \in \mathbb{Z}[X]$ of degree $n-1 > 0$ and content 1, choose some n arbitrary distinct arguments $x_1, \ldots, x_n \in \mathbb{Z}$ and multi-evaluate $y_i := p(x_i)$. Observe that, if $q \in \mathbb{Z}[X]$ is a non-trivial divisor of p, then $z_i := q(x_i)$ divides y_i for each $i = 1, \ldots, n$. This suggests to go through all (finitely many) choices for $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ with $z_i \mid y_i$, to calculate the interpolation polynomial $q \in \mathbb{Q}[X]$ to data (x_i, z_i) and check whether its coefficients are integral and q divides p.

2.2. *Undecidability of $\mathbb{A}$ in $\mathbb{R}$ with support of a $\mathbb{Q}$-oracle*

In this section, we prove $\mathbb{A} \not\preceq \mathbb{Q}$.

The undecidability of $\mathbb{A}$ *without* further oracle assistance follows similarly to that of $\mathbb{Q}$ from a continuity argument, observing that each $\mathbb{A}$ and $\mathbb{Q}$ as well as their complement are dense in $\mathbb{R}$. In fact, algebraic numbers remain dense even when restricting to arbitrary high degree:

Lemma 8. *Let $x \in \mathbb{R}$, $\varepsilon > 0$, and $N \in \mathbb{N}$. Then, there exists an algebraic real a of* $\deg(a) = N$ *with $|x - a| < \varepsilon$.*

Proof. Take some arbitrary algebraic real b of degree N, such as $b := 2^{1/N}$. Since $\mathbb{Q}$ is dense in $\mathbb{R} \ni y := x - b$, there exists some rational $r \in \mathbb{Q}$ with $|r - y| < \varepsilon$. Then $a := r + b$ has the desired property. □

Of course, total discontinuity does not prevent a problem to be BSS-decidable under the support of a $\mathbb{Q}$-oracle any more as, for example, $\mathbb{Q}$ now is decidable. More precisely a putative algorithm might try distinguishing algebraic from transcendental reals by mapping a given x through some rational function $f \in \mathbb{R}(X)$, then querying the oracle whether the value $f(x)$ is rational or not, and proceeding adaptively depending on the answer.

The following observation basically says that in any sensible such approach, for transcendental x, $f(x)$ will be irrational rather than rational.

Lemma 9. *Let $f: \mathrm{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ be analytic and non-constant, $T \subseteq \mathrm{dom}(f)$ uncountable. Then, f maps some $x \in T$ to a transcendental value, that is, $f(x) \notin \mathbb{A}$.*

Proof. Consider an arbitrary $y \in \mathbb{A}$; by uniqueness of analytic functions [26, Theorem 10.18], f can map at most countably many different $x \in \mathrm{dom}(f)$ to that single value y. Hence, if $f(x) \in \mathbb{A}$ for all $x \in T$, $f^{-1}(\mathbb{A}) = \bigcup_{y\in\mathbb{A}} f^{-1}(\{y\})$ is a countable union of countable sets and thus countable, too—contradicting the prerequisite that $T \subseteq f^{-1}(\mathbb{A})$ is uncountable. □

So it remains the case of an algorithm trying to map algebraic x to rationals $f(x)$ and transcendental x to irrational $f(x)$. The final ingredient formalizes the intuition that this approach cannot distinguish transcendentals from algebraic numbers of sufficiently high degree:

Proposition 10. *Let $f \in \mathbb{R}(X)$ be non-constant such that $f = p/q$ with polynomials p, q of* $\deg(p) < n$, $\deg(p) < m$. *Let $a_1, \dots, a_{n+m} \in \mathrm{dom}(f)$ be distinct real algebraic numbers with $f(a_1), \dots, f(a_{n+m}) \in \mathbb{Q}$.*

(a) *There are co-prime polynomials $\tilde{p}, \tilde{q}$ of $\deg(\tilde{p}) < n$, $\deg(\tilde{q}) < m$ with coefficients in the algebraic field extension $\mathbb{Q}(a_1, \dots, a_{n+m})$ such that, for all $x \in \mathrm{dom}(f) = \{x: q(x) \neq 0\} \subseteq \mathbb{R}$, it holds $f(x) = \tilde{f}(x) := \tilde{p}(x)/\tilde{q}(x)$.*
(b) *Let $d := \max_i \deg(a_i)$. Then, $f(x) \notin \mathbb{Q}$ for all transcendental $x \in \mathrm{dom}(f)$ as well as for all $x \in \mathbb{A}$ of $\deg(x) > D := d^{n+m} \cdot \max\{n-1, m-1\}$.*

Notice that p and q themselves in general do not satisfy claim (a); e.g. $p = \pi \cdot \tilde{p}$ and $q = \pi \cdot \tilde{q}$.

Proof. (a) Without loss of generality take p and q to be co-prime. Let $y_i := f(a_i)$. The idea is to solve the rational interpolation problem for (a_i, y_i). Already knowing that it *has* a solution (namely p, q) avoids many of the difficulties discussed in [21].

More precisely, observe that the coefficients $p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1} \in \mathbb{R}$ of p and q satisfy the homogeneous $(n+m) \times (n+m)$-size system of linear equations

$$\left(\begin{array}{ccccc|cccc} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & -y_1 & -y_1 a_1 & \dots & -y_1 a_1^{m-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} & -y_2 & -y_2 a_2 & \dots & -y_2 a_2^{m-1} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-1} & -y_3 & -y_3 a_3 & \dots & -y_3 a_3^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right) \cdot \begin{pmatrix} p_0 \\ \vdots \\ p_{n-1} \\ q_0 \\ \vdots \\ q_{m-1} \end{pmatrix} = 0 .$$

In particular, this system has $(p_0, \dots, q_{m-1}) \in \mathbb{R}^{n+m}$ as non-zero solution.

The coefficients of the matrix live in $\mathbb{Q}(a_1, \dots, a_{n+m})$. Therefore, Gaussian Elimination yields a (possibly different) non-zero solution $(\bar{p}_0, \dots, \bar{q}_{m-1})$, also with entries in $\mathbb{Q}(a_1, \dots, a_{n+m})$. Now apply the Euclidean Algorithm to the thus obtained polynomials $\bar{p}, \bar{q}$ and calculate their greatest common divisor $\bar{h}$ which, again, has coefficients in $\mathbb{Q}(a_1, \dots, a_{n+m})$.

Thus, $\tilde{p} := \bar{p}/\bar{h}$ and $\tilde{q} := \bar{q}/\bar{h}$ are co-prime polynomials over $\mathbb{Q}(a_1, \dots, a_{n+m})$ of $\deg(\tilde{p}) < n$ and $\deg(\tilde{q}) < m$ such that $\tilde{p} \cdot q$ coincides with $p \cdot \tilde{q}$ on arguments $a_1, \dots, a_{n+m}$. This implies the latter polynomials of degree less than $n+m$ to be identical: $\tilde{p} \cdot q = p \cdot \tilde{q}$.

It follows that q divides both sides; and co-primality of (p, q) in the factorial ring $\mathbb{R}[X]$ requires that q divides $\tilde{q}$. Similarly, $\tilde{q}$ divides q, yielding $\tilde{q} = \lambda q$ for some $\lambda \in \mathbb{R}$. Analogously, $\tilde{p} = \lambda p$ for the same λ.

(b) Consider $x \in \mathbb{R}$ with $y := f(x) \in \mathbb{Q}$ and suppose x is algebraic of $\deg(x) > d^{n+m} \cdot \max\{n-1, m-1\}$ or transcendental. Since f is non-constant, the polynomial $\tilde{p} - y \cdot \tilde{q}$ is not identically zero. Being, by virtue of (a), a zero of this polynomial with coefficients from $\mathbb{Q}(a_1, \ldots, a_n)$, x lies in an algebraic extension of the latter field, hence ruling out the case that it is transcendental. More precisely, the degree of x over $\mathbb{Q}(a_1, \ldots, a_n)$ is bounded by $\deg(\tilde{p} - y \cdot \tilde{q})$; and $\deg(x)$, its degree over $\mathbb{Q}$, is at most $\deg(\tilde{p} - y \cdot \tilde{q}) \cdot \deg(a_1) \cdots \deg(a_{n+m}) \leqslant \max\{n - 1, m - 1\} \cdot d^{n+m}$ by Eq. (1)—a contradiction. □

We are finally in the position to prove

Theorem 4. *In the* BSS *model of real number computation it holds* $\mathbb{Q} \not\preceq \mathbb{A}$. *In particular, transcendence is not semi-decidable even when using* $\mathbb{Q}$ *as an oracle set.*

Proof. Suppose some BSS algorithm semi-decides $\mathbb{T}$ in $\mathbb{R}$ with oracle $\mathbb{Q}$ according to Definition 1; in other words, it proceeds by repeatedly evaluating a given $x \in \mathbb{R}$ at functions $f \in \mathbb{R}(X)$ and continuing adaptively according to whether $f(x)$ is positive/zero/negative and rational/irrational, such as to terminate iff $x \in \mathbb{T}$.

Consider this process unrolled into an (infinite yet countable) Decision Tree, each internal node u of which is labeled with an according $f_u \in \mathbb{R}(X)$ and has five successors according to the cases

- $0 > f_u(x) \in \mathbb{Q}$
- $0 > f_u(x) \notin \mathbb{Q}$
- $0 = f_u(x)$
- $0 < f_u(x) \in \mathbb{Q}$
- $0 < f_u(x) \notin \mathbb{Q}$

with leafs corresponding to terminating computations, that is, to $x \in \mathbb{T}$. Observe that the sets T_v of $x \in \mathbb{T}$ terminating in leaf v give rise to a partition of $\mathbb{T}$. In fact, the at most countably many leafs—as opposed to $\mathbb{T}$ having cardinality of the continuum—require that T_v is uncountable for at least one v.

Consider the path leading from the root to that leaf. W.l.o.g. it contains no branches of type "$0 = f_u(x)$" nor of type "$f_u(x) \in \mathbb{Q}$" that are answered "yes"; for if it does, then the uncountable set T_v of transcendentals x passing through this branch implies that f_u is constant (Lemma 9) and node u thus is dispensable. By possibly changing from $+f_u$ to $-f_u$, we may finally suppose that every branch on the path to leaf v is of type $0 < f_u(x)$.

Summarizing, $T_v \neq \emptyset$ is the set of exactly those $x \in \mathbb{R}$ satisfying $0 < f_u(x) \notin \mathbb{Q}$ for the (finitely many) internal nodes u on the path from the root to v; in particular, $T_v \subseteq \operatorname{dom}(f_u)$. Now take some $t \in T_v \subseteq \mathbb{R}$. Due to continuity of rational functions, there exists $\varepsilon > 0$ such that $f_u(x) > 0$ on all nodes u on that path for any $x \in \mathbb{R}$ satisfying $|x - t| < \varepsilon$. In particular, $f_u(a) > 0$ holds for infinitely many algebraic numbers a of unbounded degree according to Lemma 8. Since by presumption, none of them completes the (terminating) computational path to leaf v, they must branch off somewhere, that is, satisfy $f_u(a) \in \mathbb{Q}$ for some of the finitely many nodes u. However by Proposition 10(b), each single f_u can sort out only algebraics of degree up to some finite $D = D(u)$—a contradiction. □

3. More undecidable and incomparable real degrees

A further achievement of the works of Friedberg and Muchnik was the existence of incomparable r.e. degrees below the Halting Problem. In this section, we extend our above techniques to establish in the real case such problems explicitly.

More precisely, we shall construct natural incomparable subsets of $\mathbb{A}$. They are given as certain algebraic, infinite extensions of $\mathbb{Q}$ obtained by means of adjunction of nth roots of a fixed prime.

For simplicity, we consider two incomparable problems only. However, the construction immediately generalizes to an infinite number of incomparable real r.e. Turing degrees.

3.1. Some auxiliary results from algebra

Consider the following type of algebraic extensions:

Definition 11. For fields $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$ and $0 < r \in \mathbb{Q}$, let

$$\mathbb{F}(\sqrt[*]{r}) \quad := \quad \mathbb{F}\big(\{r^{1/n} : n \in \mathbb{N}\}\big),$$

where the corresponding fractional powers are understood as positive real numbers.

Thus, $\mathbb{Q}(\sqrt[*]{2})$ results from $\mathbb{Q}$ by field adjunction of all nth roots of 2, $n \in \mathbb{N}$. The ancient proof of $\sqrt{2}$'s irrationality immediately generalizes to see that $[\mathbb{Q}(\sqrt[*]{2}) : \mathbb{Q}]$ is indeed infinite. By Lemma 12(c) below, this extends from $\mathbb{Q}$ to, e.g., $[\mathbb{Q}(\sqrt[*]{2}, \sqrt[*]{3}) : \mathbb{Q}(\sqrt[*]{2})]$. In combination with Claim (d) it generalizes Lemma 8.

Lemma 12. *(a) If* $(\frac{r}{s})^{1/n} \in \mathbb{Q}$ *for* $n \in \mathbb{N}$ *and co-prime* $r, s \in \mathbb{N}$, *then* $r^{1/n}, s^{1/n} \in \mathbb{N}$.

(b) *For* $n_1, \ldots, n_k \in \mathbb{N}$ *and squarefree* $t \in \mathbb{N}$, $\mathbb{F}\big(\sqrt[n_1]{t}, \ldots, \sqrt[n_k]{t}\big) = \mathbb{F}\big(\sqrt[N]{t}\big)$ *where* $N := \operatorname{lcm}(n_1, \ldots, n_k)$ denotes the *least common multiple*.

(c) *For distinct prime numbers* $p_1, \ldots, p_d, p_{d+1}$ *and* $n \in \mathbb{N}$, *it holds*[4]

$$\big[\mathbb{Q}\big(\sqrt[*]{p_1}, \sqrt[*]{p_2}, \ldots, \sqrt[*]{p_d}, \sqrt[*]{p_{d+1}}\big) : \mathbb{Q}\big(\sqrt[*]{p_1}, \sqrt[*]{p_2}, \ldots, \sqrt[*]{p_d}\big)\big] \quad = \quad \infty.$$

(d) *To any* $n \in \mathbb{N}$, $\varepsilon > 0$, *and* $x \in \mathbb{R}$, *there exists* $y \in \mathbb{Q}(\sqrt[*]{2})$ *of degree at least n over* $\mathbb{Q}(\sqrt[*]{3})$ *such that* $|x - y| < \varepsilon$.

Proof. (a) W.l.o.g. $n \geqslant 2$. Let $(\frac{r}{s})^{1/n} = \frac{a}{b}$ with co-prime $a, b \in \mathbb{N}$. Then any prime divisor p of s divides $s \cdot a^n = r \cdot b^n$ but not r (by co-primality) and thus b^n. Hence even p^n divides $r \cdot b^n = s \cdot a^n$, so $p^n | s$ since $p \nmid a$. This reveals that every prime factor p of s occurs in s with multiplicity a multiple of n, i.e., $s^{1/n} \in \mathbb{N}$; similarly for $r^{1/n}$.

(b) Recall that the following properties of lcm:

$$n_i | \operatorname{lcm}(n_1, \ldots, n_k) = \operatorname{lcm}\big(\operatorname{lcm}(n_1, \ldots, n_{k-1}), n_k\big)$$

and

$$\operatorname{lcm}(a, b) = ab/\gcd(a, b) = ab/(ra + st) \quad \text{with} r, s \in \mathbb{Z}.$$

Therefore, each t^{1/n_i} is a power of $t^{1/N}$ and thus in $\mathbb{F}(\sqrt[N]{t})$; while, conversely, $t^{1/\operatorname{lcm}(a,b)} = (t^{1/a})^s \cdot (t^{1/b})^r \in \mathbb{F}(\sqrt[a]{t}, \sqrt[b]{t})$ yields $t^{1/N}$ to belong to $\mathbb{F}(\sqrt[n_1]{t}, \ldots, \sqrt[n_k]{t})$ by induction on k. Hence we have indeed established $t^{1/N}$ as a primitive element.

[4] We owe considerable gratitude to Toma Albu for pointing us to [4].

(c) Besicovitch has been proven that

$$\left[\mathbb{Q}\left(\sqrt[N_1]{p_1},\ \sqrt[N_2]{p_2},\ldots,\ \sqrt[N_d]{p_d}\right):\mathbb{Q}\right] \quad = \quad N_1 \cdot N_2 \cdots N_d;$$

cf. [4, Theorem 2]; see also [1, bottom of p. 2]. Now combine with Claim (b) and Eq. (1).

(d) By (c), $b := 2^{1/n}$ has degree n over $\mathbb{Q}(\sqrt[*]{3})$; and so has $y := b + r$ for any $r \in \mathbb{Q}$. $\mathbb{Q}$ being dense, take r close to $x - b$. □

3.2. Construction of incomparable segrees

The tools from the previous subsection allow to extend our results to obtain

Theorem 13. *The sets* $\mathbb{Q}(\sqrt[*]{2})$ *and* $\mathbb{Q}(\sqrt[*]{3})$ *are recursively enumerable yet incomparable.*

Its proof is based on the following immediate generalization of Proposition 10.

Proposition 14. *Let* $f \in \mathbb{R}(X)$, $f = \frac{p}{q}$ *with polynomials* p, q *of degree less than* n *and* m, *respectively. Let* $a_1, \ldots, a_{n+m} \in \mathbb{Q}(\sqrt[*]{2}) \cap \operatorname{dom}(f)$ *be distinct with* $f(a_i) \in \mathbb{Q}(\sqrt[*]{3})$.

(*a*) *There are co-prime polynomials* $\tilde{p}, \tilde{q}$ of $\deg(\tilde{p}) < n$, $\deg(\tilde{q}) < m$ *with coefficients in the algebraic field extension* $\mathbb{Q}(\sqrt[*]{3}; a_1, \ldots, a_{n+m})$ *such that, for all* $x \in \operatorname{dom}(f) = \{x : q(x) \neq 0\} \subseteq \mathbb{R}$, *it holds* $f(x) = \tilde{f}(x) := \tilde{p}(x)/\tilde{q}(x)$.

(*b*) *Let* $d := \max_i \deg_{\mathbb{Q}(\sqrt[*]{3})}(a_i)$. *Then,* $f(x) \notin \mathbb{Q}(\sqrt[*]{3})$ *for all transcendental* $x \in \operatorname{dom}(f)$ *as well as for all* $x \in \mathbb{Q}(\sqrt[*]{2})$ *of* $\deg_{\mathbb{Q}(\sqrt[*]{3})}(x) > D := d^{n+m} \cdot \max\{n-1, m-1\}$.

Proof. [of Theorem 13] For semi-decidability observe that, by virtue of Lemma 12(b) and [20, Proposition V, §1.4], that

$$\mathbb{Q}(\sqrt[*]{2}) = \bigcup_{n\in\mathbb{N}} \mathbb{Q}[\sqrt[n]{2}] = \Big\{x \in \mathbb{Q} \,\Big|\, \exists n \in \mathbb{N}\ \exists a_0, \ldots, a_{n-1} \in \mathbb{Q}\ \underbrace{\exists y \in \mathbb{R} :\ x = a_0 + y a_1 + \cdots + y^{n-1} a_{n-1} \ \wedge\ y^n = 2}_{=:\Phi(n;a_0,\ldots,a_{n-1};x)}\Big\}.$$

Now existential quantification with respect to y amounts to an $\mathcal{NP}_{\mathbb{R}}$-formula and is decidable; cf. e.g. [2, section 2.4]. Hence membership of x to $\mathbb{Q}(\sqrt[*]{2})$ can be semi-decided by searching for $n \in \mathbb{N}$ and $a_0, \ldots, a_{n-1} \in \mathbb{Q}$.

Consider a putative machine semi-deciding $\mathbb{R} \setminus \mathbb{Q}(\sqrt[*]{2})$ by means of an $\mathbb{Q}(\sqrt[*]{3})$-oracle. Follow the proof of Theorem 4 and apply Lemma 9 to obtain in just the same way a leaf v together with the related path set $T_v \subseteq \mathbb{R} \setminus \mathbb{Q}(\sqrt[*]{2})$. Since T_v is uncountable it contains a transcendental x and in each neighborhood of x by virtue of Lemma 12(d) elements of $\mathbb{Q}(\sqrt[*]{2})$ of arbitrarily high degree over the field $\mathbb{Q}(\sqrt[*]{3})$. Thus, applying Proposition 14 there exist elements in $\mathbb{Q}(\sqrt[*]{[2]})$ that are branched along v, contradicting the assumption that the machine semi-decides $\mathbb{R} \setminus \mathbb{Q}(\sqrt[*]{2})$.

The converse claim "$\mathbb{Q}(\sqrt[*]{3}) \not\preceq \mathbb{Q}(\sqrt[*]{2})$" follows similarly. □

The numbers 2 and 3 in the above proof can obviously be replaced by any two distinct primes; that is, the sets $\mathbb{Q}(\sqrt[*]{p})$ and $\mathbb{Q}(\sqrt[*]{q})$ are incomparable for any two $p, q \in \mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$. In particular, we have *explicitly* an infinite number of incomparable degrees. Moreover the argument immediately extends to see that, for $P, Q \subseteq \mathbb{P}$,

$$\mathbb{Q}(\{\sqrt[*]{p} : p \in P\}) \preceq \mathbb{Q}(\{\sqrt[*]{q} : q \in Q\}) \iff P \subseteq Q.$$

Since the collection of subsets with inclusion is the prototype of a poset, we have thus arrived at the following.

Scholium[5] **15.** Every countable poset can be embedded into the recursively enumerable real Turing degrees. □

The latter parallels classical results in discrete recursion theory; see for instance [29, Exercise VII, §2.2(b) and Exercise VIII, §4.10].

3.3. Some open problems

The previous arguments lead to some other problems concerning the relation between some natural subsets of $\mathbb{R}$ that we consider to be interesting.

For $d \in \mathbb{N}$ let $\mathbb{A}_d := \{x \in \mathbb{A}: \deg(x) \leqslant d\} \subset \mathbb{R}$ denote the set of algebraic numbers that have degree at most d over $\mathbb{Q}$.

Problem 1. *Is it true that we have a strict chain*

$$\mathbb{Q} \precnsim \mathbb{A}_2 \precnsim \mathbb{A}_3 \precnsim \cdots \precnsim \mathbb{A} \precnsim \mathbb{H}_{\mathbb{R}} \text{ ?}$$

We have defined $\mathbb{A}_d$ to consist of numbers of degree *less or* equal to d but point out that considering, rather than $\mathbb{A}_2 =: \mathbb{A}_{\leqslant 2}$, the set $\mathbb{A}_{=2} := \{x \in \mathbb{A}: \deg(x) = 2\}$ of numbers of degree *exactly* 2, in fact makes no difference:

Lemma 15. *It holds* $\mathbb{A}_{=2} \equiv \mathbb{A}_{\leqslant 2}$.

Proof. Based on oracle access to $\mathbb{A}_{\leqslant 2}$, decide $\mathbb{A}_{=2}$ in $\mathbb{R}$ as follows: Upon input of $x \in \mathbb{R}$, query $\mathbb{A}_{\leqslant 2}$ to find out whether $\deg(x) \leqslant 2$. If not, reject; otherwise $x \in \mathbb{A}$ and we may apply Lemma 6 to compute $\deg(x)$.

Conversely, given $\mathbb{A}_{=2}$ as an oracle, decide whether $x \in \mathbb{A}_{\leqslant 2}$ by querying both x and $y := x + \sqrt{2}$. If at least one of them belongs to $\mathbb{A}_{=2}$, then x is surely algebraic and thus applicable to Lemma 6. If $x, y \in \mathbb{R} \setminus \mathbb{A}_{=2}$, we may reject immediately because $\deg(x) < 2$ would imply $x \in \mathbb{Q}$ and thus $y = x + \sqrt{2} \in \mathbb{A}_{=2}$. □

But what about this question for general degrees $d \in \mathbb{N}$?

Problem 2. *Does it hold* $\mathbb{A}_{=d} \equiv \mathbb{A}_{\leqslant d}$ *for all* $d \geqslant 2$?

Another interesting question has been kindly pointed out by a referee:

Problem 3. *Is there a countable set Turing-equivalent to the real Halting Problem* $\mathbb{H}_{\mathbb{R}}$?

[5] A scholium is "*a note amplifying a proof or course of reasoning, as in mathematics*".

A disproof of the latter would, just by reasons of cardinality, include and significantly strengthen our result $\mathbb{H}_{\mathbb{R}} \not\preceq \mathbb{Q}$ but not the stronger claim $\mathbb{A} \not\preceq \mathbb{Q}$.

4. The linear BSS model

We have so far considered the full BSS model over the reals. In the last 10 years, its linearly restricted version $(\mathbb{R}, +, -, 0, 1, <)$ has received increasing interest [11,18,24] due to its relation with the classical (i.e., discrete) "$\mathcal{P} \stackrel{?}{=} \mathcal{NP}$" question [12]. Here only additions, subtractions and comparisons as well as the constants 0 and 1 are allowed but no multiplication $\times$ nor division $\div$. Thus, all computed intermediate results on inputs $x \in \mathbb{R}$ have the form $ax + b$ for some $a, b \in \mathbb{Z}$. Analogously to the full model, the Halting Problem for linear machines is undecidable by a linear machine; and Post's Problem as well makes sense in the linear version. In order to give an explicit solution to it, we once more consider the rationals $\mathbb{Q}$, but this time as the harder of two problems. The weaker undecidable one will be the following:

Definition 16. Let $\mathbb{SQ} := \{q^2 : q \in \mathbb{Q}\}$ denote the set of quadratic rationals.

We shall show that $\mathbb{SQ} \precneqq \mathbb{Q}$, where in this section "$\preceq$" and all similar notions refer to reducibility in the linear model. We start with some easy observations. Both $\mathbb{Q}$ and $\mathbb{SQ}$ are undecidable in the linear model since this already holds in the full model. Both sets are semi-decidable: For input $x \in \mathbb{R}$ enumerate all pairs $(r, s) \in \mathbb{Z} \times \mathbb{N}$ and check for each pair whether $x \cdot s = r$. Note that both the enumeration and the "multiplication" $x \cdot s$ can be performed in $(\mathbb{R}, +, -, 0, 1, <)$; similarly for semi-deciding $\mathbb{SQ}$ by enumerating all pairs (r^2, s^2) based for instance on the recursion $(r+1)^2 = r^2 + r + r + 1$. Next, $\mathbb{SQ} \preceq \mathbb{Q}$: On input $x \in \mathbb{R}$, first check $x \geqslant 0$ and ask the $\mathbb{Q}$-oracle whether $x \in \mathbb{Q}$. If this is the case use the above enumeration to find $(r, s) \in \mathbb{N}^2$ with $xs = r$. Then test whether some of the (finitely many) pairs $(\tilde{r}^2, \tilde{s}^2) \leqslant (r, s)$ satisfies $x \cdot \tilde{s}^2 = \tilde{r}^2$ or not.

Note that in the full BSS model the converse relation $\mathbb{Q} \preceq \mathbb{SQ}$ is also valid: Having access to a $\mathbb{SQ}$-oracle one can decide $\mathbb{Q}$ by simply squaring the input $x \in \mathbb{R}$. The main result of this section reveals that this reduction does not hold in the linear model:

Theorem 17. *In the linear* BSS *model, it is* $\mathbb{SQ} \precneqq \mathbb{Q}$.

The proof applies Lemmas 18 and 19 which are in some sense linear counterparts to Proposition 10(b) and Lemma 8, respectively.

Lemma 18. *Let* $P \subseteq \mathbb{P}$ *be a (finite or infinite) set of primes. Define*

$$\mathbb{Q}_P := \left\{ \tfrac{r}{s} : r \in \mathbb{Z},\ s \in \mathbb{N} \wedge 1 = \gcd(r, s) \wedge \sqrt{s} \notin \mathbb{N} \wedge \forall p \in \mathbb{P}: (p|s \Rightarrow p \in P) \right\}$$

as the set of rationals whose denominator, in reduced form with respect to the numerator, is no square and contains only prime factors from P. This satisfies

(*a*) $\mathbb{Q}_P \cap \mathbb{SQ} = \emptyset$.
(*b*) *Let* $a \in \mathbb{Z}$ *having no prime factors P and* $b \in \mathbb{Z}$. *Then* $x \in \mathbb{Q}_P$ *implies* $y := a \cdot x + b \in \mathbb{Q}_P$.

Proof. (a) is a special case of Lemma 12(a). For (b) suppose that $x := \frac{r}{s} \in \mathbb{Q}_P$ with co-prime r, s and a, b as in the statement. Then $y = \frac{ar+bs}{s}$ with $\gcd(ar + bs, s) = 1$; the latter holds because a putative prime factor p of $\gcd(ar + bs, s)|s$ belongs to P by definition and thus does not divide a nor r, contradiction. In particular, the reduced denominator s of x is also that of y. $\square$

Lemma 19. *For each $p \in \mathbb{P}$, the set $\tilde{\mathbb{Q}}_p := \{r/p^{2k+1} : k \in \mathbb{N},\ r \in \mathbb{Z},\ p \nmid r\} \subseteq \mathbb{Q}$ is dense in $\mathbb{R}$. In particular, so is $\mathbb{Q}_P$ for any non-empty $P \subseteq \mathbb{P}$.*

Proof. The (not necessarily reduced) p-adic rationals $\mathbb{Q}_p := \{t/p^k : t \in \mathbb{Z}, k \in \mathbb{N}\}$ are obviously dense: To $x \in \mathbb{R}$ and large enough $k \in \mathbb{N}$, let $t := \lfloor x \cdot p^k \rfloor \in \mathbb{Z}$.

Now to $y = t/p^k \in \mathbb{Q}_p$ take any $n \in \mathbb{N}$ and let $r := t \cdot p^{k+2n+1} + 1$. Then $p \nmid r$, so $z := r/p^{2(k+n)+1}$ belongs to $\tilde{\mathbb{Q}}_p$; and $|z - y| = p^{-2n-2k+1}$ becomes arbitrarily small for increasing n. Hence $\tilde{\mathbb{Q}}_p$ is dense in $\mathbb{Q}_p$ and thus in turn in $\mathbb{R}$ as well.

Finally, $\mathbb{Q}_P$ is a superset of $\tilde{\mathbb{Q}}_p$ for $p \in P$. $\square$

Proof. [Theorem 17] As usual we take a potential linear $\mathbb{SQ}$-oracle machine M semi-deciding $\mathbb{R} \setminus \mathbb{Q}$ and pick a certain input $z > 0$ which this time suffices to be chosen as irrational. Let $f_i : x \mapsto a_i \cdot x + b_i$ denote the finitely many test-functions evaluated on z by M before arrival in a leaf, $a_i, b_i \in \mathbb{Z}$, $1 \leqslant i \leqslant I$. Take $P \subseteq \mathbb{P}$ such that $\mathbb{P} \setminus P$ contains all (finitely many) prime factors of these coefficients a_i and b_i. Since z is irrational, so is $f_i(z) \notin \mathbb{SQ}$ and in particular $f_i(z) \neq 0$ (w.l.o.g. > 0) for all i; hence it holds $f_i(x) > 0$ for all i and all x in some non-empty neighborhood of z. By Lemma 19 we can furthermore require $x \in \mathbb{Q}_P \subseteq \mathbb{Q}$; by Lemma 18 for this x all oracle queries "$f(x) \overset{?}{\in} \mathbb{SQ}$" are answered negatively. In other words, M branches x along the very same path as z and eventually ends up in a leaf, contradicting that M terminates only for $x \notin \mathbb{Q}$. $\square$

Problem 4. *In the linear setting, does $\mathbb{Q}$ have the same degree of undecidability as the Halting Problem?*

5. Conclusion

We have shown that oracle access to the set of rational numbers $\mathbb{Q}$ gives a BSS machine additional power but still prevents it from solving the real Halting Problem $\mathbb{H}_{\mathbb{R}}$ (of BSS machines). In addition we have explicitly specified an uncountable number of incomparable recursively enumerable degrees in the real number setting. This involved arguments from topology as well as from abstract algebra; e.g., transcendence, irreducible polynomials and finite field extensions play a major role. In the linear setting, a similar result was obtained using number theory; e.g., irrationality, primes and integral lattices.

Our proofs generally do not rely on the ordering available over the real numbers. Thus with small corrections (for example a slightly changed definition of the characteristic path in a potential decision tree) they also work over the complex numbers yielding the corresponding results.

We close with some remarks concerning hypercomputation. Since there is no commonly accepted definition of what hypercomputation should be our remarks, however, are a bit speculative. Regarding attempts to physically realize hypercomputation over the reals our results indicate that it seems advisable (since provably easier) to construct a device capable of solving $\mathbb{Q}$ rather than $\mathbb{H}_{\mathbb{R}}$. Such an approach may, in contrast to discrete hypercomputation, benefit from the explicit knowledge of this degree.

One might object that, since "*Natura non facit saltus*" according to Leibniz, the discontinuity inherent in deciding $\mathbb{Q}$ in $\mathbb{R}$ (i.e., of distinguishing fractions from general reals) makes an according devise physically impossible. However, we point out that for example the Fractional Quantum Hall Effect (Nobel Prize Physics 1998) shows that nature does exhibit exactly this kind of discontinuous behavior.

Note added in proof. We have just learned of a negative answer to Problem 4 established by C. Gassner.

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